inkMaths

ANSWER BOOK

Arc length and surface area

Further Pure 2 · Year FM

7 questions · 32 marks

NAVY EDITION

SECTION A · Warm-up

A1 $dy/dx = x^{1/2}$, so $1 + (dy/dx)^2 = 1 + x$ and the integrand is $\sqrt{1 + x}$. Integrating gives $(2/3)(1 + x)^{3/2}$, which runs from 2/3 to 16/3. The length is **14/3**, about 4.667. B1 for dy/dx, M1 for the arc length integrand, M1 for the integration, A1 for 14/3. [4]

A2 $dy/dx = \sinh x$ and $1 + \sinh^2 x = \cosh^2 x$, so the integrand is cosh x itself. The catenary is the one curve on this specification where the square root disappears without a substitution. The length is $[\sinh x]$ from 0 to 1 = **sinh** 1, about 1.175. M1 for using the identity, M1 for integrating cosh x, A1 for sinh 1. [3]

SECTION B · Standard

B1 $dx/dt = 2t$ and $dy/dt = 3t^2$, so the integrand is $\sqrt{4t^2 + 9t^4} = t\sqrt{4 + 9t^2}$ for $t \geq 0$. Substituting $u = 4 + 9t^2$ gives $(1/27)(4 + 9t^2)^{3/2}$, which runs from 8/27 to $13^{3/2}/27$. The length is $(13\sqrt{13} - 8)/27$, about **1.440**. M1 for the parametric integrand, A1 for $t\sqrt{4 + 9t^2}$, M1 for the substitution, A1 for the antiderivative, A1 for the length. [5]

B2 In polar form the integrand is $\sqrt{r^2 + (dr/d\theta)^2}$. Here $dr/d\theta = e^\theta$ as well, so the bracket is $2e^{2\theta}$ and the integrand is $\sqrt{2}\, e^\theta$. Integrating gives $\sqrt{2}(e^\pi - 1)$, about **31.31**. M1 for the polar integrand, A1 for $\sqrt{2}\, e^\theta$, M1 for integrating, A1 for the length. Every polar arc length needs r and its derivative, never r alone. [4]

B3 The surface integral is $2\pi\int y\, ds = 2\pi\int x^3\sqrt{1 + 9x^4}\, dx$ from 0 to 1. Substituting $u = 1 + 9x^4$ gives $(\pi/27)(1 + 9x^4)^{3/2}$, which runs from $\pi/27$ to $10\sqrt{10}\, \pi/27$. The area is $(\pi/27)(10\sqrt{10} - 1)$, about **3.563** square units. M1 for the surface integral, A1 for the integrand, M1 for the substitution, A1 for the antiderivative, A1 for the area. [5]

SECTION C · Stretch

C1 Since $ds = \cosh x\, dx$, the integral is $2\pi\int \cosh^2 x\, dx$ from 0 to 1. Writing $\cosh^2 x = (\cosh 2x + 1)/2$ gives $2\pi[x/2 + \sinh(2x)/4]$, so the area is $2\pi(1/2 + (\sinh 2)/4) = \pi + \pi(\sinh 2)/2$, about **8.839** square units. M1 for the surface integral, M1 for the double angle identity, A1 for the integration, M1 for the limits, A1 for the exact area. Without the double angle identity there is nothing to integrate. [5]

C2 $dr/d\theta = -2 \sin \theta$, so $r^2 + (dr/d\theta)^2 = 4(1 + \cos \theta)^2 + 4 \sin^2\theta = 8 + 8 \cos \theta = 16 \cos^2(\theta/2)$. The square root is $4|\cos(\theta/2)|$, and $\cos(\theta/2) \geq 0$ for $0 \leq \theta \leq \pi$, so the upper half has length $\int 4 \cos(\theta/2)\, d\theta = [8 \sin(\theta/2)] = 8$. By symmetry in the initial line the perimeter is **16**. M1 for $r^2 + (dr/d\theta)^2$, A1 for $16 \cos^2(\theta/2)$, B1 for the modulus and its range of validity, M1 for integrating, A1 for the half length, A1 for the perimeter. Dropping the modulus and running the integral straight from 0 to 2π returns 0, since $\cos(\theta/2)$ is negative on the second half. [6]