inkMaths

ANSWER BOOK

Centres of mass by integration

Further Mechanics 2 · Year FM

7 questions · 33 marks

NAVY EDITION

SECTION A · Warm-up

A1 $x(G) = \int xy\,dx \div \int y\,dx$ and $y(G) = \int \tfrac{1}{2}y^2\,dx \div \int y\,dx$. A vertical strip of height y acts at horizontal distance x, giving the moment $xy\,\delta x$. But it acts at its own halfway height $y/2$, so the vertical moment is $y \times y/2\,\delta x$, hence the half and the square. B1 B1 for the two integrals, B1 for the reason the vertical one carries the half and the square. Both integrals are divided by the same area, so the asymmetry is entirely in the numerators. [3]

A2 Area = $\int(4 - x^2)\,dx = 8 - 8/3 =$ **16/3**. $\int xy\,dx = \int(4x - x^3)\,dx = 8 - 4 = 4$, so $x(G) = 4 \div 16/3 =$ **0.75**. $\int \tfrac{1}{2}y^2\,dx = \tfrac{1}{2}(32 - 64/3 + 32/5) = 8.533$, so $y(G) = 8.533 \div 16/3 =$ **1.6**. B1 for the upper limit, M1 A1 for the area, M1 A1 for the x coordinate, A1 for the y coordinate. The curve meets the x-axis at $x = 2$, which is where the upper limit comes from. [6]

SECTION B · Standard

B1 By symmetry the centre of mass lies on the axis, so only $x(G)$ is needed. Volume $\propto \int y^2\,dx = \int x\,dx =$ **8**. Moment $\propto \int xy^2\,dx = \int x^2\,dx =$ **64/3**. So $x(G) = (64/3) \div 8 =$ **8/3** ≈ 2.67. B1 for the symmetry argument, M1 A1 for the volume integral, M1 for the moment integral, A1 for the position. It sits two thirds of the way along, since the solid widens towards the far end. [5]

B2 Volume $\propto \int(r^2 - x^2)\,dx$ from 0 to $r = r^3 - r^3/3 =$ **$2r^3/3$**. Moment $\propto \int x(r^2 - x^2)\,dx = r^4/2 - r^4/4 =$ **$r^4/4$**. Dividing: $x(G) = (r^4/4) \div (2r^3/3) =$ **$3r/8$**. M1 A1 for the volume integral, M1 A1 for the moment integral, A1 for the printed result. It lies well inside the flat face, because the wide end is there. [5]

B3 Each element carries a mass $\rho\,dV$ rather than dV, so the density goes inside both integrals: the total mass is $\int \rho\,dV$ and the moment is $\int x\rho\,dV$. Everything else is unchanged, since the centre of mass is still the mass-weighted average of position. M1 for the element of mass, A1 A1 for the two integrals. A constant ρ cancels top and bottom, so uniform bodies never need it. [3]

B4 Mass = $\int(1 + x)\,dx$ from 0 to $2 = 2 + 2 =$ **4** kg. Moment = $\int x(1 + x)\,dx = 2 + 8/3 =$ **14/3**. So $x(G) = (14/3) \div 4 =$ **7/6** ≈ 1.17 m. M1 A1 for the mass, M1 for the moment, A1 for the position. It sits past the midpoint, towards the denser end, which is the check to make. [4]

SECTION C · Stretch

C1 The curve and the line meet where $x^2 = 2x$, that is at $x = 0$ and $\mathbf{x} = 2$, with the line above the curve between them. [7]
Area = $\int(2x - x^2)\,dx$ from 0 to 2 = 4 - 8/3 = **4/3**.
For x(G), each vertical strip has height $2x - x^2$ and acts at distance x: $\int x(2x - x^2)\,dx = 16/3 - 4 =$ **4/3**, so x(G) = (4/3) ÷ (4/3) = **1**.
For y(G), a strip running from $y = x^2$ up to $y = 2x$ acts at the average of its ends, so its moment is $\frac{1}{2}[(2x)^2 - (x^2)^2]\,\delta x$. Then $\int\frac{1}{2}(4x^2 - x^4)\,dx = \frac{1}{2}(32/3 - 32/5) =$ **32/15**, giving y(G) = (32/15) ÷ (4/3) = **1.6**.
Centre of mass **(1,** 1.6). B1 for the upper limit, M1 A1 for the area, M1 A1 for the x coordinate, M1 A1 for the y coordinate.
The commonest loss of marks here is writing $\int\frac{1}{2}(2x - x^2)^2\,dx$ for the y moment. It is the difference of the squares that is needed, not the square of the difference.