inkMaths

ANSWER BOOK

Continuous random variables: density and distribution functions

Further Statistics 2 · Year FM

6 questions · 22 marks

NAVY EDITION

SECTION A · Warm-up

A1 Probability is the area under the density, and the area over a single point is zero, so no individual value carries any probability. It follows that $P(X < a)$ and $P(X \le a)$ are equal, so the inequality signs make no difference. B1 for the zero area over a point, B1 for the two inequalities agreeing. That is a genuine simplification compared with discrete distributions. [2]

A2 The integral of $4 - x^2$ from 0 to 2 is $[4x - x^3/3] = 8 - 8/3 = 16/3$. Setting $k \times 16/3 = 1$ gives **k** $= 3/16$. M1 for integrating and setting the total area to 1, A1 for 16/3, A1 for $k = 3/16$. The density is non-negative throughout the interval, so it is valid. [3]

SECTION B · Standard

B1 Integrate the density from the bottom of the range: $F(x) = (3/16)(4x - x^3/3) =$ $\mathbf{3x/4}$ $- x^3/16$ for $0 \le x \le 2$, with $F = 0$ below 0 and $F = 1$ above 2. A distribution function must be defined everywhere, so both of those branches are needed for full marks. Check $F(2) = 1.5 - 0.5 = 1$, which every F must satisfy at the top of its range. [4]
Then $P(X < 1) = F(1) = 0.75 - 0.0625 =$ **0.6875**. M1 for integrating the density, A1 for $3x/4 - x^3/16$, B1 for the branches outside the range, A1 for 0.6875. Reaching for the integral again wastes time once F is known.

B2 Integrating, $F(x) =$ $\mathbf{x^2/16}$ on the interval. [4]
Quartiles come from solving F = the required proportion, never from the density. The median solves $m^2/16 = 0.5$, so $m^2 = 8$ and $m = 2\sqrt{2} =$ **2.828**. The lower quartile solves $q^2/16 = 0.25$, so $q^2 = 4$ and $q =$ **2**. M1 A1 for $F(x) = x^2/16$, A1 for the median 2.828, A1 for the lower quartile 2.
Both lie inside 0 to 4, and the lower quartile lies below the median, which is the check to make before moving on. The negative roots are discarded because the density is zero below 0.

B3 Differentiating, $f(x) = x/8$ on the interval and zero elsewhere. $P(1 < X < 3) = F(3) - F(1) = 9/16 - 1/16 = 8/16 =$ **0.5**. B1 for $f(x) = x/8$, M1 for $F(3) - F(1)$, A1 for 0.5. Subtracting values of F is quicker than integrating. [3]

SECTION C · Stretch

C1 **Validity.** f is non-negative on the whole range, and the two triangles each have area 1/2, so the total area is 1. Both conditions are needed; area alone is not enough. [6]

The distribution function. For $0 \le x < 1$, $F(x) = \mathbf{x^2/2}$. For the second branch, start from the 1/2 already accumulated rather than integrating from 0, which is the step most answers get wrong: $F(x) = 1/2 + [2t - t^2/2]$ from 1 to x = $\mathbf{2x} - x^2/2 - 1$.

Two checks. At $x = 1$ both branches give 0.5, so F is continuous, and $F(2) = 4 - 2 - 1 = 1$ as it must be.

The upper quartile. Since $F(1) = 0.5 < 0.75$, the quartile lies in the **upper** branch. Choosing the branch first is worth a mark, and using $x^2/2 = 0.75$ here is the standard error.

Solve $2q - q^2/2 - 1 = 0.75$, that is $q^2 - 4q + 3.5 = 0$, giving $q = 2 \pm \sqrt{2}/2$.

The root $2 + \sqrt{2}/2 \approx 2.71$ lies outside the range, so the upper quartile is $\mathbf{q} = 2 - \sqrt{2}/2$, about 1.293. B1 for the validity check, M1 for integrating the first branch, A1 for both branches of F, B1 for choosing the upper branch, dM1 for solving the quadratic, A1 for $q = 2 - \sqrt{2}/2$. Say why the other root is rejected; a bare pair of roots does not finish the question.

