inkMaths

ANSWER BOOK

De Moivre's theorem and trigonometric identities

Complex numbers · Year FM

6 questions · 22 marks

NAVY EDITION

SECTION A · Warm-up

A1 $(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$; for a general number, raise the modulus to the power n and multiply the argument by n. The theorem speaks about moduli and arguments, so a number given as $x + yi$ must be converted first. B1 for stating the theorem, B1 for the reason. [2]

A2 $1 - i$ has modulus $\sqrt{2}$ and argument $-\pi/4$. The sixth power has modulus $(\sqrt{2})^6 = 8$ and argument $-3\pi/2$, which reduces to $\pi/2$. So $(1 - i)^6 = 8i$. B1 for the modulus and argument, M1 for applying De Moivre, A1 for 8i. Bracket-expansion agrees: $(1 - i)^2 = -2i$, and $(-2i)^3 = 8i$. [3]

SECTION B · Standard

B1 $\sqrt{3} + i$ has modulus 2 and argument $\pi/6$. The fifth power has modulus $2^5 = 32$ and argument $5\pi/6$. Converting back: $32(\cos 5\pi/6 + i\sin 5\pi/6) = 32(-\sqrt{3}/2 + i/2) =$ $\mathbf{-16\sqrt{3}}$ + 16i. B1 for the modulus and argument, M1 for the fifth power, M1 for converting back to a + bi form, A1 for $-16\sqrt{3} + 16i$. The accuracy mark goes for exact surds; a decimal such as $-27.7 + 16i$ is not the form asked for. [4]

B2 The power multiplies the argument and leaves the unit modulus alone, so the first is $\cos\pi/2 + i\sin\pi/2 =$ **i**. De Moivre holds for negative n too, so the second has argument $-\pi/2$ and equals **−i**. B1 B1, one mark each, and no working is needed for either. [2]

B3 The binomial expansion gives $\cos^3\theta + 3i\cos^2\theta\sin\theta - 3\cos\theta\sin^2\theta - i\sin^3\theta$, and De Moivre says the total is $\cos 3\theta + i\sin 3\theta$. Imaginary parts: $\sin 3\theta = 3\cos^2\theta\sin\theta - \sin^3\theta$. Replace $\cos^2\theta$ with $1 - \sin^2\theta$ to reach $\sin 3\theta = 3\sin\theta - 4\sin^3\theta$. M1 for the binomial expansion, A1 for it correct, M1 for equating imaginary parts, A1 for $\sin 3\theta = 3\cos^2\theta\sin\theta - \sin^3\theta$, A1 for the printed result. Two of the marks are for the expansion and one for saying which parts are being equated. Dropping the i from the third term is the usual slip, and it puts $3\cos\theta\sin^2\theta$ into the imaginary part by mistake. [5]

SECTION C · Stretch

C1 By De Moivre, $z^n + z^{-n} = 2\cos n\theta$, so in particular $z + 1/z = 2\cos\theta$. Raise that to the fifth power: $(z + 1/z)^5 = z^5 + 5z^3 + 10z + 10/z + 5/z^3 + 1/z^5$. [6]

Pair the terms from the ends inwards. That gives $2\cos 5\theta + 10\cos 3\theta + 20\cos\theta$, while the left side is $32\cos^5\theta$. Halving gives **16** $\cos^5\theta = \cos 5\theta + 5\cos 3\theta + 10\cos\theta$. Pairing the terms is the step the method mark is for; expanding and stopping loses it.

Now integrate the right side, which has no powers in it: $(1/16)[\sin 5\theta/5 + 5\sin 3\theta/3 + 10\sin\theta]$ from 0 to $\pi/2$. At $\pi/2$ the three sines are 1, −1 and 1, giving $(1/16)(1/5 - 5/3 + 10) = (1/16)(128/15) =$ **8/15**. At 0 everything vanishes. B1 for $z + 1/z = 2\cos\theta$, M1 for raising it to the fifth power, M1 for pairing the terms from the ends inwards, A1 for the printed identity, M1 for integrating the multiple angles, A1 for 8/15.

Turning a power into multiple angles is what makes the integral elementary; integrating $\cos^5\theta$ directly needs a substitution instead.