inkMaths

ANSWER BOOK

Disproof and proof by contradiction

Proof · Year 12–13

7 questions · 23 marks

NAVY EDITION

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SECTION A · Warm-up

A1 Try $n = 3$: $2^3 + 1 = 9 = 3 \times 3$, which is not prime. B1 for the counter example, B1 for evaluating and factorising it. One counter example, evaluated and factorised, finishes the job. The claim survives $n = 1$ and $n = 2$ (giving 3 and 5), which is exactly why it looks safe. [2]

A2 Take $a = -2$ and $b = 2$. Then $a^2 = 4 = b^2$ but $a \neq b$. B1 for the counter example, B1 for showing the squares agree while the numbers do not. Squaring throws away the sign, so equal squares only force $a = \pm b$, and any pair with opposite signs will do. A counter example has to be stated with both values evaluated; explaining in words that negatives exist is not a disproof. [2]

A3 Assume the statement is false: $\sqrt{3}$ is rational, so $\sqrt{3} = a/b$ where a and b are integers, $b \neq 0$, and the fraction is in lowest terms, meaning a and b share no common factor. B1 for assuming $\sqrt{3}$ rational and writing it as a/b, B1 for the lowest-terms condition. The lowest-terms clause is there for a reason; the final contradiction crashes into it. [2]

SECTION B · Standard

B1 Assume $\sqrt{3} = a/b$ in lowest terms. Squaring: $3 = a^2/b^2$, so $a^2 = 3b^2$. Then a^2 is divisible by 3, so a is too: write $a = 3c$. Substituting, $9c^2 = 3b^2$, so $b^2 = 3c^2$, and b is also divisible by 3. Now a and b share the factor 3, contradicting lowest terms. The assumption was false, so $\sqrt{3}$ is irrational. B1 for the assumption in lowest terms, M1 for reaching $a^2 = 3b^2$, A1 for $a = 3c$, M1 for substituting to get $b^2 = 3c^2$, A1 for the contradiction and the conclusion. [5]

B2 Assume such integers exist. The left side is $3(2a + 3b)$, a multiple of 3, so 1 would have to be a multiple of 3 as well. It is not, so the assumption fails and no such integers exist. M1 for factorising as $3(2a + 3b)$, A1 for 1 not being a multiple of 3, A1 for the conclusion. The whole method is spotting the common factor of the coefficients, and $21a + 14b = 1$ dies the same death with a factor of 7. Note that $2a + 3b$ must be declared an integer for the argument to close. [3]

SECTION C · Stretch

C1 Assume the conclusion fails: n^2 is even but n is odd, so $n = 2k + 1$. Then $n^2 = 4k^2 + 4k + 1 = 2(2k^2 + 2k) + 1$, which is odd. That contradicts n^2 being even, so n must be even. B1 for assuming n odd, M1 for squaring $n = 2k + 1$, A1 for $2(2k^2 + 2k) + 1$, A1 for the contradiction and the conclusion. This little fact is the hinge of the $\sqrt{2}$ proof, and it earns its own question for that reason. Assuming n is odd, rather than assuming the whole implication false, is the correct negation here. [4]

C2 Dividing N by any prime on the list leaves remainder 1, because the product part is exactly divisible and the + 1 is not: so none of 2, 3, 5, 7, 11, 13 divides N. But $N = 30031 = 59 \times 509$, so N is composite. M1 for the remainder argument, A1 for no listed prime dividing N, B1 for the factorisation 59×509, A1 for N being composite, B1 for the argument needing only a prime factor off the list. The argument only needs some prime factor of N to be missing from the list; here 59 and 509 both are, and either one finishes the proof. [5]