ANSWER BOOK

First order equations and integrating factors

Differential equations · Year FM

6 questions · 23 marks

NAVY EDITION

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SECTION A · Warm-up

A1 IF = $e^{\int P\,dx}$. After multiplying through, the left side is exactly the derivative of (IF × y), so one integration solves the equation. B1 for the integrating factor, B1 for the product-rule statement. [2]

A2 $\int \tan x\,dx = \ln(\sec x)$, so IF = $e^{\ln \sec x}$ = **sec** x. M1 for $\int \tan x\,dx = \ln(\sec x)$, A1 for sec x. A log-form integral collapses when it is exponentiated. Leaving the factor as $e^{\ln \sec x}$ gets the method mark but not the accuracy mark, and it makes every later line harder to handle. [2]

SECTION B · Standard

B1 Here $P = 1/x$, so IF = $e^{\ln x} = x$. Multiplying through, $d(xy)/dx = 6x^2$. [5]
Integrating gives $xy = 2x^3 + c$. The constant belongs on this line, before dividing by x; introducing it afterwards produces cx instead of c and loses the accuracy mark.
Then y = 3 at x = 1 gives 3 = 2 + c, so c = 1 and **y** = $2x^2 + 1/x$.
Check: $dy/dx + y/x = (4x - 1/x^2) + (2x + 1/x^2) = 6x$.
M1 A1 for the integrating factor, M1 for $xy = 2x^3 + c$, A1 for c = 1, A1 for the final solution.

B2 P = −2, so IF = e^{-2x} and $d(e^{-2x}y)/dx = e^{3x}$. The sign of P travels into the exponent unchanged; e^{2x} here would leave a left side that is nobody's derivative. [4]
Integrating: $e^{-2x}y = e^{3x}/3 + C$, so **y** = $e^{5x}/3 + Ce^{2x}$. M1 A1 for the integrating factor, M1 for integrating, A1 for the general solution.

B3 P = 2x is not constant, which changes nothing about the method and only the shape of the factor. IF = e^{x^2}, so $d(e^{x^2}y)/dx = 4x\,e^{x^2}$. [4]
The right side integrates by recognition, since $4x\,e^{x^2}$ is the derivative of $2e^{x^2}$. Hence $e^{x^2}y = 2e^{x^2} + C$ and **y** = $2 + Ce^{-x^2}$.
As x grows the transient Ce^{-x^2} dies away very quickly, so every solution settles onto the steady state **y** = 2, whatever value C takes. M1 for the integrating factor, A1 for e^{x^2}, A1 for the general solution, B1 for the limit.

SECTION C · Stretch

C1 $P = -2/x$, so $\int P\,dx = -2 \ln x$ and IF $= e^{-2 \ln x} = \mathbf{x^{-2}}$. Reading the factor as $e^{-2} \ln x$ without simplifying to a power of x is what makes the next integration look impossible. [6]
Multiplying through, $d(x^{-2}y)/dx = \ln x$.
Now integrate $\ln x$ by parts, taking $u = \ln x$ and $dv = dx$: $\int \ln x\,dx = x \ln x - \int 1\,dx = x \ln x - x + c$. Writing $\ln x$ as $1 \times \ln x$ is the step worth spotting; there is no integral of $\ln x$ by inspection.
So $x^{-2}y = x \ln x - x + c$, and $y = 0$ at $x = 1$ gives $0 = 0 - 1 + c$, so $c = 1$.
Hence $\mathbf{y} = x^2(x \ln x - x + 1)$, that is $x^3 \ln x - x^3 + x^2$.
Check at $x = 2$: $y = 8 \ln 2 - 8 + 4 \approx 1.545$, and differentiating the answer returns the original equation.
M1 A1 for the integrating factor, M1 for reaching $d(x^{-2}y)/dx = \ln x$, M1 A1 for the parts integration, A1 for the particular solution.