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ANSWER BOOK

Functions of two variables

Further Pure 2 · Year FM

15 questions · 64 marks

NAVY EDITION

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SECTION A · Warm-up

A1 FT-A1 [3]

Holding y constant, f_x = $\mathbf{2xy^3}$ + 5, since y^3 is a constant multiplier and the 5x differentiates to 5. Holding x constant, f_y = $\mathbf{3x^2y^2}$, and the 5x now differentiates to 0 because it carries no y. M1 for treating the other letter as a constant, A1 for f_x, A1 for f_y.

A2 FT-A2 [3]

Putting x = 2 gives the section **z** = 4 + 3y^2, a parabola opening upwards with its vertex at height 4. Putting z = 12 gives the contour $x^2 + 3y^2 = 12$, or $x^2/12 + y^2/4 = 1$, which is an **ellipse** reaching x = ±2√3 and y = ±2. B1 for the section, M1 for setting z = 12, A1 for naming the ellipse with its intercepts. A section is a curve of z against one variable; a contour is a curve in the x-y plane.

A3 FT-A3 [4]

First derivatives: $f_x = 3x^2 + 4xy$ and $f_y = 2x^2 - 4y^3$. Differentiating again gives f_{xx} = **6x** + 4y, f_{yy} = $\mathbf{-12y^2}$, and f_{xy} = **4x** from f_x by y. Differentiating f_y by x gives f_{yx} = 4x as well, so the two agree, as the mixed derivative theorem says they should. M1 for the first derivatives, A1 for f_{xx}, A1 for f_{yy}, A1 for both mixed derivatives with the comparison stated.

A4 FT-A4 [2]

Moving z across gives $x^2 + y^2 - z = 4$, so $g(x, y, z) = x^2 + y^2 - z$ and c = 4. B1 for a correct implicit form with its constant. Read as a surface, z is the height above the point (x, y), so the equation z = f(x, y) assigns one height to each point of the plane and S is the graph of that assignment. B1 for the height reading. The two forms describe the same set of points; the implicit one treats all three letters alike, which is what makes it the natural form once a surface is no longer the graph of a function, as for a sphere.

SECTION B · Standard

B1 FT-B1 [4]

$f_x = 2x - 4$ and $f_y = 2y + 6$, so both vanish only at **(2,** -3), where f = 4 + 9 - 8 - 18 + 3 = -10. Then f_{xx} = 2, f_{yy} = 2 and f_{xy} = 0, so H = 2 × 2 - 0^2 = 4. Since H > 0 and f_{xx} > 0 this is a **local** minimum, of value -10. M1 for setting both first derivatives to zero, A1 for the point, M1 for H, A1 for the classification with a reason.

B2 FT-B2 [6]

$f_x = 3x^2 - 3$ gives x = ±1 and $f_y = 3y^2 - 3$ gives y = ±1, so the stationary points are (1, 1), (1, -1), (-1, 1) and (-1, -1). Here f_{xx} = 6x, f_{yy} = 6y and f_{xy} = 0, so H = 36xy. At (1, 1), H = 36 > 0 with f_{xx} = 6 > 0: a **minimum**, value -4. At (-1, -1), H = 36 > 0 with f_{xx} = -6 < 0: a **maximum**, value 4. At (1, -1) and (-1, 1), H = -36 < 0: **saddle** points, both of value 0. M1 for solving both equations, A1 for the four points, M1 for H = 36xy, A1 A1 A1 for the three kinds of verdict. One surface can carry all three kinds at once.

B3 FT-B3

The height is $f(2, 1) = 4 - 3 = 1$, so the point is $(2, 1, 1)$. The slopes are $f_x = 2xy = 4$ and $f_y = x^2 - 6y = -2$ there. The tangent plane is $z = 1 + 4(x - 2) - 2(y - 1)$, which tidies to $\mathbf{z = 4x - 2y - 5}$. B1 for the height, M1 for both partial derivatives evaluated at the point, M1 for the formula, A1 for the plane. Checking at $(2, 1)$ returns $8 - 2 - 5 = 1$, the right height. [4]

B4 FT-B4

$f_x = \mathbf{3e^{3x}} \cos y$, with $\cos y$ a constant multiplier, and $f_y = \mathbf{-e^{3x}} \sin y$. Differentiating f_x with respect to y gives $f_{xy} = \mathbf{-3e^{3x}} \sin y$, and f_y differentiated with respect to x gives the same. At $(0, \pi/2)$ that is $-3 \times 1 \times 1 = \mathbf{-3}$. M1 A1 for f_x, A1 for f_y, A1 for the mixed derivative and its value. Exponentials and trigonometric functions obey the same rules as powers do. [4]

B5 FT-B5

Each derivative holds the other two letters constant. $f_x = \mathbf{2xy}$, since the last two terms carry no x. $f_y = \mathbf{x^2} + z^3$, since the first two terms are linear in y and the $4z$ carries no y. $f_z = \mathbf{3yz^2} + 4$. At $(1, 2, -1)$ that is $3 \times 2 \times 1 + 4 = \mathbf{10}$. M1 for holding the other letters constant, A1 for f_x, A1 for f_y, A1 for f_z with its value. Three variables need no new rule: the number of letters held fixed goes up by one and nothing else changes. Note that z^2 is positive at $z = -1$, so the answer is 10 and not -2. [4]

B6 FT-B6

Putting $x = 1$ gives $\mathbf{z} = 1 - y^2$, a parabola in the z-y plane opening downwards with its highest point at height 1. Putting $y = 0$ gives $\mathbf{z} = x^2$, a parabola opening upwards with its lowest point at the origin. B1 B1 for the two sections. Putting $z = 0$ gives $x^2 = y^2$, that is the **pair** of lines $y = x$ and $y = -x$ through the origin. B1 for the contour. Along one section the origin is the bottom of a curve and along the other it is the top, so the surface is a **saddle** there. B1 for the saddle. A section is a curve of z against one variable, so it lives on the surface; a contour is a curve in the x-y plane, and a contour that is a pair of crossing lines rather than a closed loop is itself the sign of a saddle. [4]

B7 FT-B7

By the chain rule $f_x = \mathbf{2x/(x^2} + y^2)$ and, by symmetry, $f_y = \mathbf{2y/(x^2} + y^2)$. M1 for the chain rule, A1 for both first derivatives. Differentiating f_x with respect to x by the quotient rule gives $f_{xx} = [2(x^2 + y^2) - 2x(2x)]/(x^2 + y^2)^2 = 2(y^2 - x^2)/(x^2 + y^2)^2$, and the same working in the other letter gives $f_{yy} = 2(x^2 - y^2)/(x^2 + y^2)^2$. M1 for the quotient rule. The two numerators are negatives of each other, so the sum is 0 everywhere except the origin, where f is undefined. A1 for the sum vanishing with the origin excluded. Writing down f_{yy} from f_{xx} by swapping x and y is legitimate here because the function itself is unchanged by that swap. [4]

SECTION C · Stretch

C1 FT-C1

$f_x = 4x^3 - 4y = 0$ gives $y = x^3$, and $f_y = 4y^3 - 4x = 0$ gives $x = y^3$. Substituting, $x = x^9$, so $x(x^8 - 1) = 0$ and $x = 0$, 1 or -1. The points are **(0,** 0), (1, 1) and (-1, -1). Now $f_{xx} = 12x^2$, $f_{yy} = 12y^2$ and $f_{xy} = -4$, so $H = 144x^2y^2 - 16$. At the origin $H = -16 < 0$, a **saddle** point of value 0. At (1, 1) and at (-1, -1), $H = 128 > 0$ with $f_{xx} = 12 > 0$, so both are **local** minima of value -2. M1 for the pair of equations, M1 for eliminating to $x = x^9$, A1 for the three points, M1 for H, A1 A1 for the classifications. Only real solutions count: $x^8 = 1$ contributes 1 and -1 alone. [6]

C2 FT-C2

$f_x = 4x^3$ and $f_y = 4y^3$ both vanish at the origin, so it is stationary. The second derivatives are $f_{xx} = 12x^2$, $f_{yy} = 12y^2$ and $f_{xy} = 0$, all zero at the origin, so **H** = 0 and the test decides nothing. Directly, though, $x^4 \geq 0$ and $y^4 \geq 0$ for real x and y, with equality only at $x = y = 0$, so $f > f(0, 0) = 0$ at every other point and the origin is a **minimum**. B1 for showing the point is stationary, M1 A1 for H = 0, B1 for the direct argument. A vanishing H is a gap in the test, not a verdict of its own: $x^4 - y^4$ also has H = 0 at the origin, and it has a saddle there. [4]

C3 FT-C3

At (3, 2) the temperature is T = 100 - 18 - 4 = 78, so the point is (3, 2, 78). B1 for the height. The slopes are $T_x = -4x = -12$ and $T_y = -2y = -4$ there. M1 for both partial derivatives evaluated at the point. The tangent plane is $T = 78 - 12(x - 3) - 4(y - 2)$. A1 for the plane. [6]

Substituting x = 3.1 and y = 1.9 gives 78 - 12(0.1) - 4(-0.1) = 78 - 1.2 + 0.4 = **77.2** degrees. M1 for the substitution, A1 for 77.2. Exactly, T = 100 - 2(9.61) - 3.61 = 77.17, so the estimate is **0.03** too high. B1 for the comparison.

The plane is the best flat model of the surface at that point, and the error it makes grows with the square of the step: doubling both increments to reach (3.2, 1.8) would multiply the gap by about four. Both signs matter in the substitution, since y has gone down while x has gone up.

C4 FT-C4

The surface is one base and four sides, so $S = xy + 2xz + 2yz$. The volume condition $xyz = 32$ gives $z = 32/(xy)$, and substituting turns $2xz$ into $64/y$ and $2yz$ into $64/x$, so $S = xy + 64/x + 64/y$. B1 for the substitution eliminating z. [6]

Then $S_x = y - 64/x^2$ and $S_y = x - 64/y^2$. Setting both to zero gives $y = 64/x^2$ and $x = 64/y^2$; substituting the first into the second gives $x = 64x^4/4096 = x^4/64$, so $x^4 = 64x$ and $x(x^3 - 64) = 0$. Since x is a length, x = 4, and then y = 4 and z = 32/16 = 2. M1 for both partial derivatives, M1 for solving them together, A1 for **4** cm by 4 cm by 2 cm, with S = 16 + 16 + 16 = **48** cm^2. A1 for the area.

For the nature, $S_{xx} = 128/x^3 = 2$, $S_{yy} = 128/y^3 = 2$ and $S_{xy} = 1$, so $H = 2 \times 2 - 1 = 3 > 0$ with $S_{xx} > 0$, a local minimum. B1 for the Hessian test with its verdict.

The root x = 0 is thrown out by the context rather than by the algebra, and a box with no lid is not a cube: the height comes out at half the side of the base, because the missing top means the vertical faces are cheaper than they would otherwise be.

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