inkMaths

ANSWER BOOK

Hyperbolic functions and identities

Hyperbolic functions · Year FM

7 questions · 25 marks

NAVY EDITION

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SECTION A · Warm-up

A1 $\cosh x = (e^x + e^{-x})/2$ and $\sinh x = (e^x - e^{-x})/2$. At zero, $\cosh 0 = 1$ and $\sinh 0 = 0$, mirroring $\cos 0$ and $\sin 0$. B1 for the two definitions, B1 for both values at zero. [2]

A2 $e^{\ln 2} = 2$ and $e^{-\ln 2} = 1/2$, so $\cosh(\ln 2) = (2 + 1/2)/2 = 5/4$ and $\sinh(\ln 2) = (2 - 1/2)/2 = 3/4$. M1 for both exponential values, A1 for 5/4, A1 for 3/4. As a check, $(5/4)^2 - (3/4)^2 = 1$. [3]

SECTION B · Standard

B1 $\operatorname{arsinh} x = \ln(x + \sqrt{x^2 + 1})$, so $x = \ln(2 + \sqrt{5}) \approx 1.444$. sinh is one-to-one, so this is the only solution. M1 for the logarithmic form of arsinh, A1 for $\ln(2 + \sqrt{5})$, B1 for the uniqueness. No ± appears here, unlike the cosh case in the next question. [3]

B2 $\operatorname{arcosh}(5/4) = \ln(5/4 + \sqrt{25/16 - 1}) = \ln(5/4 + 3/4) = \ln 2$. cosh is even, so $x = \pm\ln 2$. M1 for the logarithmic form, A1 for ln 2, A1 for the second root, B1 for the evenness argument. Every value of cosh above 1 is hit twice, once on each side of the valley. [4]

B3 Divide the whole identity by $\cosh^2 x$: $1 - \tanh^2 x = \operatorname{sech}^2 x$. M1 for dividing through by $\cosh^2 x$, A1 for the identity. The trig counterpart is $1 + \tan^2 x = \sec^2 x$; the sign flip on the $\tanh^2$ term is Osborn's rule at work, since $\tanh^2$ contains a product of two sinh terms. [2]

B4 Substitute: $2 + 2\sinh^2 x - \sinh x = 2$, so $\sinh x\,(2\sinh x - 1) = 0$. Either $\sinh x = 0$, giving $x = 0$, or $\sinh x = 1/2$, giving $x = \ln(1/2 + \sqrt{5/4}) = \ln((1 + \sqrt{5})/2)$. M1 for the substitution, A1 for the factorised quadratic, B1 for $x = 0$, M1 for the logarithmic form, A1 for $\ln((1 + \sqrt{5})/2)$. Converting $\cosh^2$ into $\sinh^2$ first turns the equation into a quadratic in one variable. [5]

SECTION C · Stretch

C1 Go back to the definitions. The left side is $5(e^x + e^{-x})/2 - 4(e^x - e^{-x})/2 = (e^x + 9e^{-x})/2$, so the equation becomes $e^x + 9e^{-x} = 6$. Writing $u = e^x$ and multiplying by u gives $u^2 - 6u + 9 = 0$, that is $(u - 3)^2 = 0$. The repeated root $u = 3$ gives $x = \mathbf{ln}\ 3$ and nothing else. Geometrically the curve $y = 5\cosh x - 4\sinh x$ has minimum value 3, since $\sqrt{5^2 - 4^2} = 3$, so the line $y = 3$ touches it rather than cutting it. M1 for using the exponential definitions, A1 for $e^x + 9e^{-x} = 6$, M1 for the quadratic in u, A1 for the repeated root, A1 for ln 3, B1 for the reason there is only one solution. Substituting u for e^x beats hunting for an identity here, because $5\cosh x - 4\sinh x$ has no neat single-function form. [6]