InkMaths

ANSWER BOOK

Implicit and parametric differentiation

Differentiation · Year 12–13

8 questions · 33 marks

NAVY EDITION

inkphysics.co.uk/maths/ · youtube.com/@InkPhysics

SECTION A · Warm-up

A1 y^3 gives $3y^2$ dy/dx. Differentiate as though y were the variable, then attach dy/dx by the chain rule. The product xy needs the product rule, giving y + x dy/dx. B1 B1, one for each result, and these two moves are the whole of implicit differentiation. [2]

A2 Differentiating both sides with respect to x gives 2y dy/dx = 4, so dy/dx = 2/y. At (4, 4) the gradient is 2/4 = ½. M1 for the implicit step, A1 for the gradient. Notice the answer depends only on y, so the curve has two different gradients above and below the x-axis at the same value of x. [2]

SECTION B · Standard

B1 Differentiate every term where it stands: 2x + 2y dy/dx = 0, so dy/dx = -x/y and at (6, 8) the gradient is -6/8 = -3/4. M1 for the implicit differentiation, A1 for 2x + 2y dy/dx = 0, M1 for the rearrangement, A1 for -3/4. As a check, the radius to (6, 8) has gradient 8/6 = 4/3, and the two gradients multiply to -1, so the calculus agrees with the perpendicular-radius geometry. [4]

B2 Differentiating term by term gives 4x + 6y dy/dx - (y + x dy/dx) = 0, where the bracket comes from the product rule on xy. Gathering the dy/dx terms: dy/dx (6y - x) = y - 4x, so dy/dx = (y - 4x)/(6y - x). At (2, 2), which does lie on C since 8 + 12 - 4 = 16, the gradient is (2 - 8)/(12 - 2) = -6/10 = -3/5. M1 for the implicit differentiation, A1 for the product-rule term, M1 for gathering, A1 for the expression, A1 for -3/5. Treating xy as a single power and writing its derivative as x dy/dx alone throws away the A mark and every mark after it. [5]

B3 dy/dx = (dy/dt)/(dx/dt) = $3t^2/(2t)$ = 3t/2, valid for t ≠ 0. At t = 2 the point is (4, 8) and the gradient is 3. M1 for both derivatives, A1 for 3t/2, B1 for the point, A1 for the gradient. One clock drives both coordinates and the gradient is one rate divided by the other, so no Cartesian conversion is needed at any stage. [4]

SECTION C · Stretch

C1 Differentiating gives $2x + (y + x\, dy/dx) + 2y\, dy/dx = 0$, so $dy/dx\,(x + 2y) = -(2x + y)$ and $dy/dx = -(2x + y)/(x + 2y)$. A tangent parallel to the x-axis needs $dy/dx = 0$, so the numerator must vanish: $2x + y = 0$, that is $y = -2x$. Substituting into the equation of C gives $x^2 - 2x^2 + 4x^2 = 12$, so $3x^2 = 12$ and $x = \pm 2$. The points are $(2, -4)$ and $(-2, 4)$. M1 implicit differentiation, A1 correct, M1 rearrange for dy/dx, A1 expression, M1 set the numerator to zero and substitute, A1 both points. Setting the whole fraction to zero by clearing the denominator instead is the wrong instinct, since it is the numerator alone that controls a zero gradient. The denominator vanishing would give a vertical tangent, which is the question you get asked next. [6]

C2 $dx/dt = -3 \sin t$ and $dy/dt = 2 \cos t$, so $dy/dx = -(2 \cos t)/(3 \sin t)$. At $t = \pi/3$ this is $-(2 \times \tfrac{1}{2})/(3 \times \sqrt{3}/2) = -2/(3\sqrt{3}) = -2\sqrt{3}/9$. The point is $(3 \cos \pi/3, 2 \sin \pi/3) = (3/2, \sqrt{3})$. The tangent is $y - \sqrt{3} = -(2/(3\sqrt{3}))(x - 3/2)$. Multiplying through by $3\sqrt{3}$ gives $3\sqrt{3}\, y - 9 = -2x + 3$, so $2x + 3\sqrt{3}\, y = 12$. M1 for both derivatives, M1 for the ratio, A1 for the gradient, B1 for the point, M1 for the straight-line equation, A1 for the final form. Check by substituting the point back: $3 + 3\sqrt{3} \times \sqrt{3} = 3 + 9 = 12$. Rationalising the gradient before forming the equation saves most of the algebraic mess at the end. [6]

C3 Differentiate with respect to y rather than x, giving $dx/dy = \cos y$. Turning the rate upside down, $dy/dx = 1/\cos y$. Now $\cos y = \sqrt{(1 - \sin^2 y)} = \sqrt{(1 - x^2)}$, where the positive root is correct because $\cos y \geq 0$ on the stated range. Hence $dy/dx = 1/\sqrt{(1 - x^2)}$. M1 for dx/dy, M1 for the reciprocal, M1 for the Pythagorean replacement, A1 for the conclusion. The stated range is not decoration. Without it the square root could take either sign, and this is the derivative of arcsin obtained by running a rate backwards. [4]