inkMaths

ANSWER BOOK

Induction: divisibility and matrices

Further proof · Year FM

7 questions · 30 marks

NAVY EDITION

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SECTION A · Warm-up

A1 $f(1) = 8 - 1 = 7$, which is divisible by 7. Hypothesis: assume $f(k) = 8^k - 1$ is divisible by 7 for some positive integer k. B1 for the base case, B1 for the hypothesis. [2]

A2 $f(k + 1) - 8f(k) = (8^{k+1} - 1) - (8^{k+1} - 8) = 7$. So $f(k + 1) = 8f(k) + 7$, a multiple of 7 plus 7, and so divisible by 7. With the base case, 7 divides $8^n - 1$ for all n. M1 for forming $f(k + 1) - 8f(k)$, A1 for the constant remainder, A1 for the conclusion. [3]

SECTION B · Standard

B1 Base: $3^2 + 7 = 16 = 8 \times 2$. Step: with $f(k) = 3^{2k} + 7$ assumed divisible by 8, $f(k + 1) = 9 \times 3^{2k} + 7 = 9f(k) - 56$. Both terms are multiples of 8 ($56 = 8 \times 7$), so $f(k + 1)$ is too. B1 for the base case, M1 for writing $f(k + 1)$ in terms of $f(k)$, A1 for $9f(k) - 56$, A1 for both terms being multiples of 8, A1 for the conclusion. The multiplier 9 comes from $3^{2(k+1)} = 9 \times 3^{2k}$: each step multiplies the power part by 9. [5]

B2 Base: M^1 has rows (1, 1), (0, 1), matching n = 1. Step: assume M^k has rows (1, k), (0, 1). Then $M^{k+1} = M^kM$ multiplies to rows (1, k + 1), (0, 1), since the top-right entry is $1 \times 1 + k \times 1 = k + 1$ and the rest are unchanged. So the pattern holds for all n. B1 for the base case, M1 for writing $M^{k+1} = M^kM$, A1 for the top-right entry, A1 for the required form, A1 for the conclusion. The shear just accumulates its slide. [5]

B3 Base: D^1 has rows (2, 0), (0, 3) = (2^1, 0), (0, 3^1). Step: assume the result at k; multiplying by D scales the top-left entry by 2 and the bottom-right by 3, giving 2^{k+1} and 3^{k+1} with the zeros preserved. B1 for the base case, M1 for multiplying by D, A1 for the top-left entry, A1 for the bottom-right entry, A1 for the conclusion. Diagonal matrices power entry by entry, and induction is how that sentence is earned. [5]

B4 Choose m to match the growth of the power term, so the powers cancel. For $8^n - 1$ take m = 8, for $3^{2n} + 7$ take m = 9. What remains is a constant, and if that constant is divisible by the target then $f(k + 1) = m f(k) +$ constant inherits divisibility from the hypothesis. B1 for choosing m to match the growth of the power term, B1 for the powers cancelling to leave a constant, B1 for that constant carrying the divisibility. [3]

SECTION C · Stretch

C1 Base: at $n = 1$ the formula gives rows $(3, -4)$ and $(1, -1)$, which is M. Step: assume M^k has rows $(1 + 2k, -4k)$ and $(k, 1 - 2k)$. Then $M^{k+1} = M^kM$, and the four entries are $3(1 + 2k) - 4k = 3 + 2k$, $-4(1 + 2k) + 4k = -4 - 4k$, $3k + (1 - 2k) = k + 1$ and $-4k - (1 - 2k) = -1 - 2k$. Rewriting, $3 + 2k = 1 + 2(k + 1)$, $-4 - 4k = -4(k + 1)$ and $-1 - 2k = 1 - 2(k + 1)$, so the rows are $(1 + 2(k + 1), -4(k + 1))$ and $(k + 1, 1 - 2(k + 1))$, the formula at $n = k + 1$. True at $n = 1$ and inherited at every step, so true for all positive integers n. B1 for the base case, M1 for forming M^kM, M1 for at least one correct entry, A1 A1 for all four entries, A1 for rewriting them in terms of $k + 1$, A1 for the conclusion. Write M^{k+1} as M^kM rather than MM^k and the hypothesis sits ready on the left; either order works here, but matrices do not commute in general, so the product is worth writing down explicitly rather than assuming. [7]