inkMaths

ANSWER BOOK

Integration by substitution and by parts

Integration · Year 12–13

8 questions · 32 marks

NAVY EDITION

SECTION A · Warm-up

A1 The bracket's derivative, $2x$, is sitting alongside as a factor, so the integrand is exactly what the chain rule leaves behind. The integral is $(x^2 + 1)^4/4 + c$, the next power up divided by that power. M1 for the recognition, A1 for the answer including the constant. [2]

A2 The numerator is exactly the derivative of the denominator, so the f'/f pattern applies and the integral is $\ln(x^2 + 3) + c$. M1 A1. No modulus signs are needed, since $x^2 + 3$ is positive for every real x. [2]

SECTION B · Standard

B1 From $u = x^2 + 1$, $du = 2x\,dx$, so $x\,dx = du/2$. The limits convert as well: $x = 0$ gives $u = 1$ and $x = 2$ gives $u = 5$. The integral becomes $\frac{1}{2}\int_1^5 u^3\,du = \frac{1}{2}[u^4/4]$ from 1 to 5 $= (625 - 1)/8 = 78$. M1 for du/dx, M1 for the substitution, A1 for the integral fully in u with converted limits, M1 for integrating, A1 for 78. Carrying the old limits 0 and 2 into a u-integral gives $(16 - 1)/8$ and is the commonest way to lose the last three marks. [5]

B2 Choose the factor that simplifies when differentiated, so $u = x$ and $dv/dx = e^{2x}$. Then $\int xe^{2x}\,dx = (x/2)e^{2x} - \int(1/2)e^{2x}\,dx = (x/2)e^{2x} - e^{2x}/4 + c$. M1 for a correct application of the formula, A1 for the first stage, M1 for integrating the remaining term, A1 for the answer with its constant. The trade turned a product into a bare exponential, and choosing which factor plays which role is the whole judgement of the method. [4]

B3 Here the logarithm has to be the differentiated factor, because $\int \ln x\,dx$ is not a standard result you can quote. Take $u = \ln x$ and $dv/dx = x$. Then $\int x \ln x\,dx = (x^2/2)\ln x - \int(x^2/2)(1/x)\,dx = (x^2/2)\ln x - x^2/4 + c$. M1 for the choice and the formula, A1 for the first stage, M1 for simplifying $x^2/2 \times 1/x$ to $x/2$, A1 for the answer. Whenever a logarithm turns up in a parts question it takes the u role, without exception at this level. [4]

B4 Parts with $u = x$ and $dv/dx = \sin x$ gives $[-x\cos x]_0^{\pi/2} + \int_0^{\pi/2}\cos x\,dx = [-x\cos x + \sin x]$ from 0 to $\pi/2$. At the upper limit that is $-(\pi/2)(0) + 1 = 1$, and at the lower limit it is $0 + 0 = 0$, so the value is exactly 1. M1 for the parts, A1 for $-x\cos x + \sin x$, M1 for substituting the limits, A1 for 1. The boundary term vanishes at both ends for different reasons, $\cos(\pi/2) = 0$ at the top and $x = 0$ at the bottom, and losing the sign on $-\int(-\cos x)\,dx$ is the usual slip. [4]

SECTION C · Stretch

C1 From $u = 1 + \sqrt{x}$, $\sqrt{x} = u - 1$ and so $x = (u - 1)^2$, giving $dx = 2(u - 1)\,du$. The limits move too: $x = 1$ gives $u = 2$ and $x = 4$ gives $u = 3$. The integral becomes $\int_2^3 2(u - 1)/u\,du = 2\int_2^3 (1 - 1/u)\,du = 2[u - \ln u]$ from 2 to 3 $= 2[(3 - \ln 3) - (2 - \ln 2)] = 2[1 - \ln(3/2)] = 2 - 2\ln(3/2)$, as required. M1 for x in terms of u, M1 for dx, A1 for the new limits, M1 for the integrand fully in u, M1 for integrating, A1 for the printed form. Splitting $(u - 1)/u$ into $1 - 1/u$ is the step that makes it integrable. Left as a single fraction it goes nowhere. [6]

C2 Parts with $u = x^2$ and $dv/dx = e^x$ gives $x^2e^x - 2\int x e^x\,dx$. The remaining integral needs parts a second time, giving $\int x e^x\,dx = x e^x - e^x$. So the antiderivative is $x^2e^x - 2x e^x + 2e^x$. At $x = 1$ that is $e - 2e + 2e = e$, and at $x = 0$ it is $0 - 0 + 2 = 2$, so the value is $e - 2$. M1 for the first parts, A1 for the first stage, M1 for the second parts, A1 for the full antiderivative, A1 for $e - 2$. Each application drops the power on x by one, so a polynomial of degree n needs n rounds and the sign alternates as you go. [5]