InkMaths

ANSWER BOOK

Leibnitz's theorem and the Weierstrass substitution

Further Pure 1 · Year FM

7 questions · 29 marks

NAVY EDITION

SECTION A · Warm-up

A1 The nth derivative of fg is the sum over r from 0 to n of ${}^nC_r$ times the rth derivative of f times the (n − r)th derivative of g. B1 for the sum with binomial coefficients, B1 for the orders of the two derivatives. The coefficients are binomial because each differentiation chooses which factor to act on. [2]

A2 $\sin x = 2t/(1 + t^2)$, $\cos x = (1 - t^2)/(1 + t^2)$ and $dx = 2\,dt/(1 + t^2)$. B1 B1 B1, one for each. The dx conversion is the one most often forgotten, and it changes every answer. [3]

SECTION B · Standard

B1 Take $f = x^3$, whose derivatives are $3x^2$, $6x$, 6 and then zero, and $g = e^x$, which never changes. Only four terms survive: $x^3 + {}^4C_1(3x^2) + {}^4C_2(6x) + {}^4C_3(6)$, all multiplied by e^x. [4]

With ${}^4C_1 = 4$, ${}^4C_2 = 6$ and ${}^4C_3 = 4$, that is $\mathbf{e^x(x^3 + 12x^2 + 36x + 24)}$. M1 for the derivatives of x^3, M1 for the Leibnitz sum, A1 for the binomial coefficients, A1 for the answer. Differentiating four times by hand gives the same polynomial with far more writing and far more chance of a slip.

B2 $1 + \cos x = 1 + (1 - t^2)/(1 + t^2) = 2/(1 + t^2)$, so the integrand times dx becomes $(1 + t^2)/2 \times 2\,dt/(1 + t^2) = dt$. [4]

The limits transform too: $x = 0$ gives $t = 0$, and $x = \pi/2$ gives $t = \tan(\pi/4) = 1$. Carrying the old limits through is the standard error.

The integral is $[t]$ from 0 to 1 = **1**. M1 for substituting for cos x and dx, A1 for the integrand reducing to dt, B1 for the transformed limits, A1 for the value. The substitution reduced the whole problem to integrating 1.

B3 $\sec x\,dx = (1 + t^2)/(1 - t^2) \times 2\,dt/(1 + t^2) = 2\,dt/(1 - t^2)$. [5]

Partial fractions: $2/((1 - t)(1 + t)) = 1/(1 - t) + 1/(1 + t)$, which integrates to $-\ln|1 - t| + \ln|1 + t| = \mathbf{\ln|(1 + t)/(1 - t)|} + c$. Note the minus sign from the chain rule on the first term; dropping it inverts the answer.

At $x = \pi/3$, $t = \tan(\pi/6) = 1/\sqrt{3}$, and at $x = 0$, $t = 0$, where the bracket is $\ln 1 = 0$.

So the integral is $\ln((1 + 1/\sqrt{3})/(1 - 1/\sqrt{3})) = \ln((\sqrt{3} + 1)/(\sqrt{3} - 1))$. Rationalising by multiplying top and bottom by $\sqrt{3} + 1$ gives $(4 + 2\sqrt{3})/2$, so the answer is $\mathbf{\ln(2 + \sqrt{3})}$, about 1.317. M1 for the substitution, A1 for $2\,dt/(1 - t^2)$, M1 for the partial fractions giving the printed form, B1 for the transformed limits, A1 for the exact value.

SECTION C · Stretch

C1 Take $f = x^2$ and $g = e^{2x}$. Then $f' = 2x$, $f'' = 2$ and every later derivative of f is zero, while the kth derivative of g is $2^k e^{2x}$. [5]

Only three terms of the Leibnitz sum survive, those with r = 0, 1 and 2:

r = 0: ${}^nC_0\, x^2 \cdot 2^n e^{2x} = 2^n x^2 e^{2x}$.

r = 1: ${}^nC_1\, (2x) \cdot 2^{n-1} e^{2x} = n \cdot 2x \cdot 2^{n-1} e^{2x} = n2^n x e^{2x}$.

r = 2: ${}^nC_2\, (2) \cdot 2^{n-2} e^{2x} = [n(n - 1)/2] \cdot 2 \cdot 2^{n-2} e^{2x} = n(n - 1)2^{n-2} e^{2x}$.

Adding gives the stated result. M1 for the derivatives of f and g, M1 for the Leibnitz sum stopping at r = 2, A1 A1 A1 for the three terms. Check it at n = 2: $e^{2x}(4x^2 + 8x + 2)$, which is what two direct differentiations produce.

The whole method rests on f being a polynomial. Its third derivative is zero, so the sum stops after three terms however large n becomes.

C2 With $\cos x = (1 - t^2)/(1 + t^2)$: [6]

$5 - 3\cos x = [5(1 + t^2) - 3(1 - t^2)]/(1 + t^2) = (2 + 8t^2)/(1 + t^2)$. Put the whole denominator over one fraction before inverting it; handling the 5 and the cosine separately is where this goes wrong.

So the integrand times dx becomes $(1 + t^2)/(2 + 8t^2) \times 2\,dt/(1 + t^2) = 2\,dt/(2 + 8t^2) =$ **$dt/(1 + 4t^2)$**. The $(1 + t^2)$ cancels, which is the point of the substitution.

Limits: x = 0 gives t = 0 and x = π/2 gives t = 1.

$\int dt/(1 + 4t^2) = \tfrac{1}{2}\arctan(2t)$, since the inner 2t contributes a factor of 2 to be divided out.

Evaluating: $\tfrac{1}{2}\arctan 2 - \tfrac{1}{2}\arctan 0 =$ **$\tfrac{1}{2}$** $\arctan 2$, about 0.554. M1 for substituting for cos x, A1 for the single fraction, M1 for forming the integrand in t, A1 for $dt/(1 + 4t^2)$, B1 for the transformed limits, A1 for the exact value. Forgetting the ½ is the last mark, and it is the one most often lost.