InkMaths

ANSWER BOOK

Limits and L'Hospital's rule

Further Pure 1 · Year FM

7 questions · 27 marks

NAVY EDITION

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SECTION A · Warm-up

A1 If f and g both tend to 0, or both tend to infinity, as x approaches a, then the limit of f/g equals the limit of f'/g', provided that second limit exists. Top and bottom are differentiated separately. B1 for the rule, B1 for the 0/0 or ∞/∞ condition. The quotient rule plays no part. [2]

A2 Both parts vanish at 0, so apply the rule, giving sin x/(2x), still 0/0. A second application gives cos x/2 → 1/2. M1 for the first application, M1 for the second, A1 for the limit. Alternatively cos x = 1 - x^2/2 + ..., so the numerator is x^2/2 + ... and the quotient tends to 1/2 in one line. [3]

SECTION B · Standard

B1 Series: e^x = 1 + x + x^2/2 + ..., so the numerator is x^2/2 + x^3/6 + ... and the quotient tends to 1/2. L'Hospital: (e^x - 1)/(2x), still 0/0, then e^x/2 → 1/2. Both routes agree. M1 for the exponential series, A1 for the numerator, A1 for the limit, M1 for two applications of the rule, A1 for confirming it. The series route needed no differentiation at all. [5]

B2 The numerator is x^3/3 + 2x^5/15 + ..., so dividing by x^3 gives 1/3 + 2x^2/15 + ... → 1/3. M1 for using the given series, A1 for the quotient, A1 for the limit. L'Hospital needs three rounds here, each one differentiating $\sec^2$x again. Once a series is handed to you in the question, use it. [3]

B3 Take logarithms: ln y = x ln(1 + 3/x) = ln(1 + 3/x)/(1/x), which is 0/0 as x → ∞. Applying L'Hospital, or expanding ln(1 + u) ≈ u - u^2/2, gives ln y → 3. So the limit is e^3 ≈ 20.09. M1 for taking logarithms, M1 for writing it as a quotient, M1 for applying the rule, A1 for ln y → 3, A1 for the limit. Power forms must be logged before the rule can see a quotient. [5]

B4 L'Hospital differentiates numerator and denominator separately, giving (1 + cos x)/(1 + $\sec^2$x) → 2/2 = 1. The quotient rule computes the derivative of the whole fraction, which answers a different question entirely. The correct limit is 1, and the series route agrees, since both parts are 2x + ... near zero. B1 for naming the misuse of the quotient rule, M1 for differentiating numerator and denominator separately, A1 for the limit. [3]

SECTION C · Stretch

C1 As it stands $x \ln x$ is of the form $0 \times (-\infty)$, which L'Hospital cannot touch, so rewrite it as the quotient $x \ln x = \ln x/(1/x)$, now of the form $-\infty/\infty$. Differentiating top and bottom gives $(1/x)/(-1/x^2) = -x$, which tends to 0. So $x \ln x \to 0$. For the second part let $y = x^x$. Then $\ln y = x \ln x \to 0$, and since the exponential is continuous, $y \to e^0 =$ **1**. M1 for rewriting the product as a quotient, M1 for applying the rule, A1 for the first limit, M1 for taking logarithms of x^x, A1 for $\ln y \to 0$, A1 for the second limit. Two moves make this work. A product is turned into a quotient so the rule applies, and the power is logged before any limit is taken. Quoting $0^0 = 1$ as a reason begs the question, because 0^0 is itself indeterminate. [6]