inkMaths

ANSWER BOOK

Locating roots and iteration

Numerical methods · Year 13

8 questions · 22 marks

NAVY EDITION

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SECTION A · Warm-up

A1 $f(1) = 1 + 1 - 5 = -3 < 0$ and $f(2) = 8 + 2 - 5 = 5 > 0$. There is a change of sign, and f is continuous on [1, 2], so a root lies in the interval. M1 for evaluating f at both ends, A1 for the sign change, continuity and conclusion. Both halves of that sentence carry marks. Candidates who compute the two values but never state the sign change or the continuity lose the conclusion mark. [2]

A2 The function might be discontinuous. Take $f(x) = 1/x$, which is negative at $x = -1$ and positive at $x = 1$ yet never equals zero, because of the asymptote at $x = 0$. The argument needs f to be continuous across the whole interval before the sign change proves anything. B1 for a counterexample with a discontinuity, B1 for continuity as the missing condition. [2]

A3 From $x^3 + x - 5 = 0$, isolate the cube to get $x^3 = 5 - x$, then take the cube root of both sides to get $x = (5 - x)^{1/3}$. M1 for isolating the cube, A1 for the printed form. Any correct isolation earns the marks, though not every rearrangement produces an iteration that converges. [2]

SECTION B · Standard

B1 $x_1 = (5 - 1.5)^{1/3} = 3.5^{1/3} = 1.51829$. $x_2 = (5 - 1.51829)^{1/3} = 1.51564$. $x_3 = (5 - 1.51564)^{1/3} = 1.51603$. The iterates close in on each other while alternating either side of the root. M1 for a correct first iterate, A1 for x_2, A1 for x_3. Feed the calculator's full unrounded value into the next step rather than the printed 5 decimal places, or the last digit will drift. [3]

B2 Test the ends of the interval that rounds to 1.516. With $f(x) = x^3 + x - 5$, $f(1.5155) = -0.0038\ldots < 0$ and $f(1.5165) = 0.0041\ldots > 0$. The sign change, together with the continuity of f, traps the root in (1.5155, 1.5165), and every number in that interval rounds to 1.516. M1 for testing the ends of the rounding interval, A1 for both values, A1 for the sign change and conclusion. Testing 1.515 and 1.517 instead is the standard mistake, since that interval is twice as wide and proves nothing about the third decimal place. [3]

B3 The gradient of g is small there. $g'(\alpha) \approx -0.145$, and an iteration of the form $x = g(x)$ converges near a root whenever $|g'| < 1$. B1 for the gradient condition, B1 for its value at the root. The negative sign is what makes the path spiral in from alternate sides rather than climb a staircase from one side. [2]

SECTION C · Stretch

C1 $x_1 = 5 - 1.5^3 = 5 - 3.375 = 1.625$, and $x_2 = 5 - 1.625^3 = 0.70898...$. The values then swing ever further from the root, reaching 4.64 and then -95. Here $g(x) = 5 - x^3$ has $g'(x) = -3x^2$, which is about -6.9 near the root, and $|g'| > 1$ throws the iterates away. M1 for a correct iterate, A1 for both values, M1 for differentiating g, A1 for the size of the gradient explaining the divergence. A valid rearrangement is not always a usable one, and the size of g′ decides which. [4]

C2 If $x_n \to \alpha$ then, g being continuous, taking limits on both sides of $x_{n+1} = g(x_n)$ gives $\alpha = g(\alpha)$, which rearranges back to $f(\alpha) = 0$. Settling digits show only that successive iterates agree with each other, and a slowly converging iteration can hold four decimal places steady for several steps while still being wrong in the fourth. The accuracy claim needs the sign-change test on the rounding interval of the claimed value. M1 for taking limits on both sides, A1 for $\alpha = g(\alpha)$, B1 for successive iterates only agreeing with each other, B1 for the sign-change test being needed. [4]