inkMaths

ANSWER BOOK

Maclaurin series

Further algebra and series · Year FM

7 questions · 27 marks

NAVY EDITION

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SECTION A · Warm-up

A1 $f(x) = f(0) + f'(0)x + f''(0)x^2/2! + f'''(0)x^3/3! + \ldots$: the coefficient of x^r is the rth derivative at zero divided by r factorial. B1 for the expansion, B1 for the general coefficient. [2]

A2 The derivatives at 0 cycle 1, 0, -1, 0, 1, so $\cos x = 1 - x^2/2! + x^4/4! + \ldots = 1 - x^2/2 + x^4/24 + \ldots$. M1 for the derivative values at 0, A1 for the x^2 term, A1 for the x^4 term. Only even powers survive, matching cos being an even function. [3]

SECTION B · Standard

B1 $1 - 0.02 + 0.0000667 = 0.980067$, and the true value is 0.980067 to six decimal places. The next term is $0.2^6/720$, about 9×10^{-8}. M1 for substituting into the three terms, A1 for 0.980067, B1 for the comment on accuracy. Three terms are tight here only because 0.2 is small; at $x = 2$ the same three terms give -0.333, against a true value of -0.416. [3]

B2 Substitute 3x into the exponential series: $1 + 3x + 9x^2/2 + 27x^3/6 = 1 + 3x + (9/2)x^2 + (9/2)x^3 + \ldots$. The x^3 coefficient is $27/3! = 9/2$. M1 for substituting 3x, A1 for the x^2 term, A1 for the x^3 term, A1 for the coefficient 9/2. Substitution into a standard series beats fresh differentiation every time it is available. [4]

B3 $\ln(1 + x)$ converges for $-1 < x \le 1$. Substituting 3x means the substituted quantity must stay in that window: $-1 < 3x \le 1$, so $-1/3 < x \le 1/3$. B1 for the original window, M1 for substituting 3x, A1 for the new window. The window transforms with the same substitution as the series. [3]

SECTION C · Stretch

C1 Subtracting the two log series cancels the even powers and doubles the odd ones: $2(x + x^3/3 + x^5/5 + \ldots)$. At $x = 1/3$ the argument is $(4/3)/(2/3) = 2$, and two terms give $2(1/3 + 1/81) = 0.6914$, against $\ln 2 = 0.6931$. M1 for subtracting the two series, A1 for the even powers cancelling, A1 for the odd-power form, M1 for putting $x = 1/3$, A1 for the argument being 2, A1 for 0.6914. Far faster than the plain $\ln(1 + x)$ series at $x = 1$, which needs hundreds of terms for the same accuracy. [6]

C2 Two standard series stack here. Write $u = \sin x = x - x^3/6 + \ldots$, then $\ln(1 + u) = u - u^2/2 + u^3/3 - \ldots$. Work to x^3 throughout, so $u^2 = x^2 + \ldots$ with the next correction at x^4, and $u^3 = x^3 + \ldots$. Substituting, $\ln(1 + \sin x) = (x - x^3/6) - x^2/2 + x^3/3 + \ldots = \mathbf{x} - x^2/2 + x^3/6 + \ldots$. M1 for putting $u = \sin x$, M1 for the $\ln(1 + u)$ expansion, A1 for u^2 to the required order, A1 for u^3, M1 for substituting and collecting, A1 for the final series. Differentiating four times instead is legitimate but long; the second derivative of $\ln(1 + \sin x)$ alone is a quotient with $\sin x$ and $\cos^2 x$ in it. Decide how far you need each series before substituting, or the x^3 coefficient picks up terms that should have been discarded. [6]