



ANSWER BOOK

Maximum flow and the labelling procedure

Decision Mathematics 2 · Year FM

7 questions · 26 marks

NAVY EDITION

SECTION A · Warm-up

- A1** One is the **spare** capacity, written as an arrow in the direction of the arc: how much more can be pushed forward. The other is the **flow** already there, written as an arrow against the arc: how much could be taken back. BI BI for the two labels. Both are needed, because an augmenting path may reduce the flow on an arc as well as increase it. [2]
- A2** It carries **4** units, the smallest spare capacity along the path. The arc that had spare capacity 4 becomes **saturated**, so no later path can use it in the forward direction. BI for 4 units, BI for the arc becoming saturated. [2]

SECTION B · Standard

- B1** S-A-C-T has spare capacities 12, 7 and 10, so push **7** and saturate AC. S-A-D-T now has 5, 6 and 11 available, so push **5**, which uses up SA. S-B-D-T has 9, 8 and 6 left, so push **6**, saturating DT. Total flow = $7 + 5 + 6 = 18$. MI for a flow-augmenting path with its flow, AI for 7, AI for 5, AI for 6, AI for the total 18, BI for showing no augmenting path remains. The arcs then carry SA 12, SB 6, AC 7, AD 5, BD 6, CT 7, DT 11. Check conservation before stopping: A takes in 12 and sends out $7 + 5$, B takes in 6 and sends out 6, C takes 7 and sends 7, D takes $5 + 6$ and sends 11. No augmenting path remains. From S only SB has spare capacity, from B only BD, and D's only outlet DT is full. [6]
- B2** Take the source side to be **{S, A, B, D}**, everything still reachable from S by an augmenting step. The arcs from that set to **{C, T}** are **AC** and **DT**, of capacities 7 and 11, so the cut has capacity **18**. Both are carrying their full capacity, and BD, which runs inside the source side, is not counted. A flow of 18 and a cut of 18 agree, so by the **max-flow** min-cut theorem the flow is maximal and this cut is minimal. MI for choosing a cut, AI for naming AC and DT, AI for the capacity 18, BI for quoting max-flow min-cut. State the arcs the cut crosses; a bare number identifies no cut. [4]
- B3** An augmenting path may run backwards along an arc that already carries flow, reducing it there so that more can travel by another route. A path can therefore still exist even when every direct route is blocked by a full arc. Only the absence of any flow-augmenting path, forwards or backwards, settles the matter, and a cut of matching capacity is what proves it. BI for the backward step, BI for the correct stopping condition. [2]

- B4** Begin with **4** units already on BD, so that arc opens with spare capacity $8 - 4 = 4$ and potential backflow 4, and no augmenting path is ever allowed to take the backflow below the required minimum. [3]
- The maximum is still **18**. Every flow of 18 must send 7 along CT and 11 along DT, and since AD carries at most 6, BD must carry at least $11 - 6 = 5$. The minimum of 4 was already being met, so it costs nothing. B1 for starting with 4 units on BD, M1 for showing BD must carry at least 5, A1 for 18 being attainable.

SECTION C · Stretch

- C1** The student's observation is correct as far as it goes. S-A-T is blocked by SA, [7]
S-A-B-T by SA, and S-B-T by BT. No forward path survives.
- Now allow a backward step. On the path **S-B-A-T** the labels are: SB has spare capacity $5 - 3 = 2$; AB is traversed **against** its direction, so the label to use is the flow on it, 3; AT has spare capacity $4 - 3 = 1$. The path therefore carries **1** unit, limited by AT.
- Update: SB rises to 4, AB falls to 2, AT rises to 4. The flow is now SA 6, SB 4, AB 2, AT 4, BT 6, of value **10**. Conservation holds at A (6 in, 2 + 4 out) and at B (4 + 2 in, 6 out).
- To prove 10 is maximal, take the cut with source side **{S, A, B}**. It crosses **AT** and **BT** only, of capacities 4 and 6, giving **10**. Flow and cut agree, so by the max-flow min-cut theorem the flow is maximal.
- M1 for finding a path with a backward step, A1 for the three labels, A1 for 1 unit, A1 for the new flow of 10, M1 for choosing a cut, A1 for the cut capacity 10, B1 for quoting max-flow min-cut.
- The unit sent backwards along AB did not vanish. It was rerouted: B now draws that unit from S instead of from A, which frees A to send it straight to T.