inkMaths

ANSWER BOOK

Mean, variance and skewness of continuous variables

Further Statistics 2 · Year FM

7 questions · 28 marks

NAVY EDITION

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SECTION A · Warm-up

A1 $E(X) = \int x^2/8\,dx = [x^3/24]$ from 0 to 4 $= 64/24 =$ **8/3**, about 2.667. $E(X^2) = \int x^3/8\,dx = [x^4/32] = 256/32 = 8$. So $\mathrm{Var}(X) = 8 - (8/3)^2 = 8 - 64/9 =$ **8/9**, about 0.889. M1 for the integrals, A1 for 8/3, A1 for 8/9. [3]

A2 $E(3X + 2) = 3E(X) + 2 = 3 \times 8/3 + 2 =$ **10**. $\mathrm{Var}(3X + 2) = 3^2\mathrm{Var}(X) = 9 \times 8/9 =$ **8**. B1 for 10, M1 for using $a^2\mathrm{Var}(X)$, A1 for 8. The shift changes the mean but not the spread, and the scale factor squares in the variance. [3]

SECTION B · Standard

B1 $F(x) = x^2/16$, so the median solves $m^2/16 = 0.5$, that is $m^2 = 8$ and $m = 2\sqrt{2} =$ **2.828**. The density increases throughout the range, so the mode sits at the right-hand end, **4**. [4]

The mean is $8/3 = 2.667$. Since mean $2.667 <$ median $2.828 <$ mode 4, the distribution is **negatively** skewed, with its tail running to the left. M1 for solving $F(m) = 0.5$, A1 for 2.828, B1 for the mode 4, B1 for the skew with its justification. Quote the three values in order; the skew verdict rests on that comparison.

B2 $E(X) = 3\int(x - 2x^2 + x^3)\,dx = 3(1/2 - 2/3 + 1/4) = 3/12 =$ **0.25**. [6]

$E(X^2) = 3\int(x^2 - 2x^3 + x^4)\,dx = 3(1/3 - 1/2 + 1/5) = 3/30 = 0.1$, so $\mathrm{Var}(X) = 0.1 - 0.25^2 = 0.1 - 0.0625 =$ **0.0375**.

Integrating the density, $F(x) = 1 - (1 - x)^3$. The median solves $(1 - m)^3 = 0.5$, so $1 - m = 0.7937$ and $m =$ **0.206**.

The density falls throughout, so the mode is at **0**. Then mode $0 <$ median $0.206 <$ mean 0.25, so the distribution is **positively** skewed. M1 for the mean integral, A1 for 0.25, M1 for $E(X^2)$ with the variance formula, A1 for 0.0375, A1 for the median 0.206, B1 for the skew. Expanding the bracket before integrating is safer than trying to integrate $3(1 - x)^2 x$ directly.

B3 $E(1/X) = \int (1/x)(x/8)\,dx$ from 0 to 4 $= \int 1/8\,dx = [x/8] =$ **0.5**. M1 for the integral of $(1/x)f(x)$, A1 for the simplified integrand, A1 for 0.5. This is not $1/E(X)$, which would be $3/8 = 0.375$. The expectation of a function is not the function of the expectation. [3]

B4 $E(aX + b) = \int(ax + b)f(x)\,dx = a\int xf(x)\,dx + b\int f(x)\,dx = aE(X) + b$, using the fact that the density integrates to 1. Then $\mathrm{Var}(aX + b) = E([aX + b - aE(X) - b]^2) = E(a^2[X - E(X)]^2) = a^2E([X - E(X)]^2) = a^2\mathrm{Var}(X)$. M1 for $E(aX + b) = aE(X) + b$, M1 for substituting into the variance definition, A1 for $a^2\mathrm{Var}(X)$. The constant b disappears because it shifts the variable and its mean equally, leaving the deviations untouched. [3]

SECTION C · Stretch

C1 $F(x) = x^2/16$ on the range, so each quartile solves $x^2/16 = p$. [6]

Q_1: $x^2 = 4$, so $Q_1 =$ **2**.

Q_2: $x^2 = 8$, so $Q_2 = 2\sqrt{2} =$ **2.828**.

Q_3: $x^2 = 12$, so $Q_3 = 2\sqrt{3} =$ **3.464**.

Then $Q_3 - 2Q_2 + Q_1 = 3.464 - 5.657 + 2 = -0.193$, and $Q_3 - Q_1 = 1.464$. The quartile coefficient is $-0.193/1.464 =$ **-0.132** to three decimal places.

For the other measure, $\sigma = \sqrt{(8/9)} = 0.943$, so $3(\text{mean} - \text{median})/\sigma = 3(2.667 - 2.828)/0.943 =$ **-0.515**.

M1 for solving $F(x) = p$, A1 for all three quartiles, M1 for the quartile coefficient, A1 for -0.132, B1 for -0.515, B1 for the comparison.

The two disagree in size but agree in sign, and the sign is what carries the meaning: both report **negative** skew, matching mean < median < mode found earlier. They differ because the quartile measure reads only the middle half of the distribution and is scaled by the interquartile range, while the other uses the whole distribution through σ. Comparing the numerical values of two different coefficients of skewness is the error to avoid; only the sign transfers between them.