inkMaths

ANSWER BOOK

Modelling with differential equations

Differential equations · Year FM

6 questions · 25 marks

NAVY EDITION

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SECTION A · Warm-up

A1 The y term is the restoring force pulling back towards equilibrium, its coefficient the stiffness; the y' term is resistance, draining energy in proportion to speed. B1 B1 for the two terms. [2]

A2 Discriminant $64 - 100 = -36 < 0$: light damping. The roots are $-4 \pm 3i$, so the motion oscillates at angular frequency 3 inside a decaying envelope e^{-4t}: frequency from the imaginary part, decay from the real part. M1 for the discriminant, A1 for light damping, B1 for the frequency and decay rate. [3]

SECTION B · Standard

B1 Discriminant $100 - 100 = 0$: critical damping, repeated root $m = -5$, solution $y = (A + Bt)e^{-5t}$. Critical damping returns the system to rest as fast as possible with no overshoot; lighter damping bounces, heavier creeps. M1 for the discriminant, A1 for critical damping, A1 for the general solution, B1 for the design reason. [4]

B2 Discriminant $25 - 16 = 9 > 0$: heavy damping, real roots -1 and -4. The solution $Ae^{-t} + Be^{-4t}$ creeps back to equilibrium without ever oscillating, the slower exponential setting the pace at late times. M1 for the discriminant, A1 for heavy damping with the roots, B1 for the long-term description. [3]

B3 Differentiate the first equation: $x'' = 2\,dy/dt = -4x$, so $x'' + 4x = 0$ and $x = A\cos 2t + B\sin 2t$. $x(0) = 3$ gives $A = 3$; $y = x'/2 = -A\sin 2t + B\cos 2t$ at $t = 0$ gives $B = 0$. So $x = 3\cos 2t$ and $y = -3\sin 2t$: the pair orbit a circle of radius 3 at angular speed 2. M1 for differentiating to eliminate y, A1 for $x'' + 4x = 0$, A1 for the general solution, M1 for applying both conditions, A1 for x and y. [5]

SECTION C · Stretch

C1 **Complementary** function. The auxiliary equation $m^2 + 2m + 5 = 0$ has discriminant $4 - 20 = -16$, so $m = -1 \pm 2i$ and the complementary function is $e^{-t}(A \cos 2t + B \sin 2t)$. [8]

Particular integral. Try $y = p \cos t + q \sin t$. Then $y' = -p \sin t + q \cos t$ and $y'' = -p \cos t - q \sin t$. Substituting and collecting:
$(-p + 2q + 5p)\cos t + (-q - 2p + 5q)\sin t = 8\cos t$, so $4p + 2q = 8$ and $4q - 2p = 0$.
From the second, $p = 2q$; then $8q + 2q = 8$ gives $q =$ **0.8** and $p =$ **1.6**.
General solution. $y = e^{-t}(A \cos 2t + B \sin 2t) + 1.6\cos t + 0.8\sin t$.
Conditions. Apply them to the general solution, never to the complementary function alone; that is the error that ends this question. $y(0) = 0$ gives $A + 1.6 = 0$, so $A =$ **−1.6**.
Differentiating, $y'(0) = -A + 2B + 0.8$, so $1.6 + 2B + 0.8 = 0$ and $B =$ **−1.2**.
Hence $\mathbf{y} = -e^{-t}(1.6\cos 2t + 1.2\sin 2t) + 1.6\cos t + 0.8\sin t$.
M1 for the auxiliary equation, A1 for the complementary function, M1 for a suitable particular integral, A1 for p and q, M1 for applying the conditions to the general solution, A1 for the value of A, A1 for the value of B, B1 for the steady state.
Long term. The e^{-t} factor kills the first part, which is the **transient**. What remains is the **steady** state $1.6\cos t + 0.8\sin t$: an oscillation at the driving frequency, of amplitude $\sqrt{(1.6^2 + 0.8^2)} = \sqrt{3.2} \approx$ **1.79**. The system ends up oscillating at the frequency it is driven at, not at its own natural frequency of 2, and the initial conditions leave no trace.