inkMaths

ANSWER BOOK

Newton's laws with a variable force

Further Mechanics 2 · Year FM

6 questions · 24 marks

NAVY EDITION

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SECTION A · Warm-up

A1 dv/dt and $v\,dv/dx$, which are equal by the chain rule. Use dv/dt when the force depends on time or on velocity and time is wanted; use $v\,dv/dx$ when it depends on position, or on velocity and distance is wanted. The aim is an equation that separates into two variables. B1 for the two forms, B1 for the rule for choosing. [2]

A2 The force depends on x, so use $3v\,dv/dx = 12 - 3x$, that is $v\,dv/dx = 4 - x$. Integrating: $\tfrac{1}{2}v^2 = 4x - \tfrac{1}{2}x^2 + c$, and $v = 0$ at $x = 0$ gives $c = 0$. So $v^2 = 8x - x^2$, and at $x = 2$, $v^2 = 16 - 4 = 12$, giving $v =$ **3.46** m/s. M1 for $v\,dv/dx = 4 - x$, A1 for $\tfrac{1}{2}v^2 = 4x - \tfrac{1}{2}x^2$ with $c = 0$, A1 for 3.46 m/s. [3]

SECTION B · Standard

B1 Again the force depends on x: $2v\,dv/dx = 6/(x + 1)^2$. Integrating: $v^2 = \int 6/(x + 1)^2\,dx = -6/(x + 1) + c$. With $v = 0$ at $x = 0$, $c = 6$, so $v^2 = 6 - 6/(x + 1)$. At $x = 2$: $v^2 = 6 - 2 = 4$ and $v =$ **2** m/s. M1 for $2v\,dv/dx = 6/(x + 1)^2$, A1 for the integral, M1 for finding c, A1 for 2 m/s. However far it travels the speed cannot exceed $\sqrt{6}$, since the force dies away too fast. [4]

B2 At the surface the force per unit mass is g, so $GM = gR^2$ and at distance x it is gR^2/x^2. Work $= \int gR^2/x^2\,dx$ from R to $3R$ $= gR^2[-1/x] = gR^2(1/R - 1/(3R)) =$ **$2gR/3$**. With Earth values that is $2(9.8)(6.4 \times 10^6)/3 \approx$ **41.8** MJ per kilogram. B1 for $GM = gR^2$, M1 for integrating gR^2/x^2 from R to $3R$, A1 for $2gR/3$, A1 for the numerical value. That is two thirds of the $gR \approx 62.7$ MJ per kilogram needed to escape completely. [4]

B3 Those equations are derived by integrating a constant acceleration, so they are simply not solutions of the problem when a varies. Using the average force gives the right total impulse only if the force is averaged over time, and the right work only if it is averaged over distance, and those are different averages. Neither gives the correct answer for both. B1 for the derivation assuming constant acceleration, B1 for the time average giving impulse, B1 for the distance average giving work. [3]

SECTION C · Stretch

C1 At distance x from the centre the acceleration is $-gR^2/x^2$, and distance is wanted, so use $v\,dv/dx = -gR^2/x^2$. Integrating gives $\tfrac{1}{2}v^2 = gR^2/x + c$. [8]
To escape, v must remain positive however large x becomes, and the limiting case has v tending to 0 as x grows without bound, so $c = 0$. At the surface $\tfrac{1}{2}v^2 = gR$, giving $v = $ **$\sqrt{(2gR)}$**. Substituting, $\sqrt{(2 \times 9.8 \times 6.4 \times 10^6)} =$ **11200** m/s, about 11.2 km/s.
For the second body, $v^2 = 2gR/4 = gR/2$ at $x = R$. Then $\tfrac{1}{2}(gR/2) = gR + c$ gives $c = -3gR/4$. The body stops where $v = 0$, that is $gR^2/x = 3gR/4$, so $x =$ **4R/3** from the centre and it rises **R/3** above the surface, about 2130 km for the Earth.
M1 for $v\,dv/dx = -gR^2/x^2$, A1 for $\tfrac{1}{2}v^2 = gR^2/x + c$, M1 for $c = 0$ in the limiting case, A1 for $\sqrt{(2gR)}$, A1 for 11200 m/s, M1 for $v^2 = gR/2$ at the surface, A1 for the stopping distance from the centre, A1 for the rise above the surface.
Half the escape speed carries a quarter of the energy and buys only a third of a radius. The mass of the body plays no part in either result.

