inkMaths

ANSWER BOOK

Numerical methods for differential equations

Further Pure 1 · Year FM

7 questions · 25 marks

NAVY EDITION

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SECTION A · Warm-up

A1 Forward: $(y_{n+1} - y_n)/h$. Central: $(y_{n+1} - y_{n-1})/(2h)$. The central difference is the more accurate for the same step length, because it straddles the point rather than looking only ahead. B1 for the two differences, B1 for naming the central one with a reason. [2]

A2 First step: $y_1 = 1 + 0.1(0 + 1) = 1.1$. Second step: $y_2 = 1.1 + 0.1(0.01 + 1.1) = 1.211$. M1 for the Euler step, A1 for y_1, A1 for y_2. Each step reuses the previous value, so an early rounding error propagates through everything after it. [3]

SECTION B · Standard

B1 The approximation is $(y_{n+1} - 2y_n + y_{n-1})/h^2$. Rearranged: $y_{n+1} = 2y_n - y_{n-1} + h^2$ times the second derivative at x_n. B1 for the approximation, M1 for rearranging, A1 for the result. Two previous values are needed, That is the reason a second order problem always supplies two starting conditions. [3]

B2 $h = 0.25$ and the ordinates are 1, 0.939413, 0.778801, 0.569783, 0.367879. The weighted sum with pattern 1, 4, 2, 4, 1 is 8.962265, and multiplying by $h/3$ gives 0.7469. B1 for $h = 0.25$, M1 for the ordinates, M1 for the 1, 4, 2, 4, 1 weighting, A1 for 0.7469. The true value is 0.7468, so four strips are accurate to three decimal places on a function with no elementary antiderivative. [4]

B3 The rule fits a parabola through each consecutive group of three ordinates, which uses up two strips at a time. An odd number leaves one strip with no partner and no parabola to fit, so the rule cannot be applied as it stands. B1 for a parabola through each group of three ordinates, B1 for strips being used two at a time. [2]

B4 Rearranged, $y_{n+1} = 2y_n - y_{n-1} + h^2(2x_n - y_n)$. [4]
At $n = 1$, with $x_1 = 0.1$: $y_2 = 2(1.05) - 1 + 0.01(0.2 - 1.05) = 1.1 - 0.0085 =$ **1.09150**.
At $n = 2$, with $x_2 = 0.2$: $y_3 = 2(1.0915) - 1.05 + 0.01(0.4 - 1.0915) = 1.133 - 0.006915 =$ **1.12609**.
M1 for the recurrence, M1 for the first step, A1 for y at 0.2, A1 for y at 0.3.
The recurrence needs the two values before it, so a second order problem always supplies two starting conditions. Keep five decimal places through the working; rounding y_2 to 1.09 shifts y_3 in the third place.

SECTION C · Stretch

C1 The improved Euler step is $y_{n+1} = y_n + \frac{1}{2}(k_1 + k_2)$, with $k_1 = hf(x_n, y_n)$ and $k_2 = hf(x_n + h, y_n + k_1)$. The second increment is evaluated at the **predicted** point, not at the current one. [7]

First step, from (0, 1):

$k_1 = 0.1(0^2 + 1) = \mathbf{0.1}$.

$k_2 = 0.1(0.1^2 + 1.1) = 0.1(1.11) = \mathbf{0.111}$.

$y_1 = 1 + \frac{1}{2}(0.1 + 0.111) = \mathbf{1.1055}$.

Second step, from (0.1, 1.1055):

$k_1 = 0.1(0.01 + 1.1055) = \mathbf{0.11155}$.

$k_2 = 0.1(0.04 + 1.21705) = \mathbf{0.125705}$.

$y_2 = 1.1055 + \frac{1}{2}(0.11155 + 0.125705) = 1.1055 + 0.1186275 = \mathbf{1.22413}$ to five decimal places.

Comparison. The exact value is $3e^{0.2} - 0.04 - 0.4 - 2 = \mathbf{1.22421}$. The improved Euler estimate is out by about **0.00008**; the forward difference value 1.211 is out by about **0.0132**, roughly 160 times worse.

M1 for k_1 and k_2 with k_2 at the predicted point, A1 for k_2, A1 for y_1, M1 for the second step, A1 for 1.22413, B1 for the exact value, B1 for the comparison of errors.

The forward difference uses the gradient at the left end of each interval and so lags a curve that is bending upwards. Averaging the gradients at the two ends removes most of that lag, at the cost of one extra evaluation of the gradient per step.