inkMaths

ANSWER BOOK

Probability generating functions

Further Statistics 1 · Year FM

6 questions · 25 marks

NAVY EDITION

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SECTION A · Warm-up

A1 $G(t) = E(t^X) = \Sigma\, t^x P(X = x)$. Setting $t = 1$ makes every power 1, so $G(1)$ is the total probability, which is always 1. B1 for the definition, B1 for $G(1) = 1$. It is the quickest check on any generating function you write down. [2]

A2 This is the binomial form $(q + pt)^n$, so **X** ~ B(8, 0.4). Differentiating, $G'(t) = 8(0.4)(0.6 + 0.4t)^7$, and at $t = 1$ the bracket is 1, so $E(X) = G'(1) =$ **3.2**. B1 for naming the binomial, M1 for differentiating and setting $t = 1$, A1 for 3.2. That agrees with $np = 8 \times 0.4$. Quote $E(X) = G'(1)$ explicitly, since the mark is for the method rather than the number. [3]

SECTION B · Standard

B1 $G''(t) = 8(7)(0.4)^2(0.6 + 0.4t)^6$, so $G''(1) = 56 \times 0.16 = 8.96$. Then $\mathrm{Var}(X) = G''(1) + G'(1) - [G'(1)]^2 = 8.96 + 3.2 - 10.24 = 1.92$, which matches $npq = 8(0.4)(0.6)$. M1 for the second derivative, A1 for $G''(1) = 8.96$, M1 for the variance formula, A1 for 1.92. [4]

B2 $G(t) = \Sigma\, t^x e^{-\lambda}\lambda^x/x! = e^{-\lambda}\Sigma(\lambda t)^x/x! = e^{-\lambda}e^{\lambda t} = e^{\lambda(t - 1)}$. Differentiating: $G'(t) = \lambda e^{\lambda(t - 1)}$, so $G'(1) = \lambda$. M1 for the defining sum, A1 for pulling out $e^{-\lambda}$, A1 for $e^{\lambda(t - 1)}$, M1 for differentiating, A1 for $G'(1) = \lambda$. Checking $G(1) = e^0 = 1$ confirms the derivation. [5]

B3 $G_{X+Y}(t) = e^{2(t - 1)} \times e^{7(t - 1)} = e^{9(t - 1)}$, which is the generating function of Po(9). Since generating functions determine distributions uniquely, X + Y ~ Po(9). M1 for multiplying the two generating functions, A1 for $e^{9(t - 1)}$, A1 for Po(9). [3]

SECTION C · Stretch

C1 By the quotient rule $G'(t) = p/(1 - qt)^2$, so $G'(1) = p/p^2 =$ **1/p**. Differentiating again, $G''(t) = 2pq/(1 - qt)^3$, so $G''(1) = 2pq/p^3 =$ **2q/p²**. [8]
$\mathrm{Var} = G''(1) + G'(1) - [G'(1)]^2 = 2q/p^2 + 1/p - 1/p^2 = (2q + p - 1)/p^2$. Since $p - 1 = -q$ the numerator is $2q - q = q$, giving **q/p²** as required.
For independent variables the generating functions multiply, so the sum of three has **G(t)** $= [pt/(1 - qt)]^3$. Differentiating a product of three identical factors gives three times the single mean, so the mean is **3/p**, and independence lets the variances add, giving **3q/p²**.
M1 for $G'(t)$ by the quotient rule, A1 for $1/p$, M1 for $G''(t)$, A1 for $2q/p^2$, M1 for the variance formula, A1 for q/p^2, B1 for the cubed generating function, B1 for the mean and variance of the sum. With $p = 0.25$ the single distribution has mean 4 and variance 12, and the third success is expected on trial 12 with variance 36. The multiplication rule holds only because the trials are independent, and saying so is part of the answer.