



ANSWER BOOK

Roots of polynomials

Further algebra and series · Year FM

7 questions · 30 marks

NAVY EDITION

SECTION A · Warm-up

- A1** Divide by 2 first: $z^3 + 3z^2 - 2z + 5 = 0$. Then $\alpha + \beta + \gamma = -3$ and $\alpha\beta\gamma = -5$. B1 for the [2]
sum of the roots, B1 for the product. Skipping the divide is the standard way to lose both marks.
- A2** $\alpha + \beta = 6$ and $\alpha\beta = 4$ straight from the coefficients. Then $1/\alpha + 1/\beta = (\alpha + [3]
\beta)/(\alpha\beta) = 6/4 = 3/2$. B1 for the sum and product, M1 for the common denominator, A1 for $3/2$. Combine over a common denominator and both known quantities appear.

SECTION B · Standard

- B1** $\alpha^2 + \beta^2 = (\alpha + \beta)^2 - 2\alpha\beta = 36 - 8 = 28$, and $\alpha^2\beta^2 = (\alpha\beta)^2 = 16$. A quadratic with [4]
those roots is $z^2 - 28z + 16 = 0$, that is z^2 minus (sum) z plus product, with the symmetric functions doing all the work. M1 for $(\alpha + \beta)^2 - 2\alpha\beta$, A1 for 28, B1 for $\alpha^2\beta^2 = 16$, A1 for the equation.
- B2** $2^3 - 16 + 2 + 6 = 0$, so $z - 2$ is a factor and the cubic is $(z - 2)(z^2 - 2z - 3) = (z - [5]
2)(z - 3)(z + 1)$, roots 2, 3, -1. Sum: $4 = -(-4)$; product: $-6 = -6/1$. M1 for substituting $z = 2$, A1 for the value zero, M1 for factorising the cubic, A1 for the other two roots, B1 for the check. Both relations hold, a worthwhile check on the factorising.
- B3** Substitute $w = z + 1$, so $z = w - 1$: $(w - 1)^2 - 6(w - 1) + 4 = w^2 - 8w + 11 = 0$. [4]
Alternatively the new sum is $6 + 2 = 8$ and the new product $\alpha\beta + \alpha + \beta + 1 = 4 + 6 + 1 = 11$. Both routes give $z^2 - 8z + 11 = 0$. M1 for the substitution $w = z + 1$, dM1 for expanding, A1 for the new sum 8, A1 for the new product 11.
- B4** Read the coefficients off with alternating signs, as $z^3 - (\text{sum})z^2 + (\text{pairs})z - [3]
(\text{product})$. That gives $z^3 + 0z^2 - 2z + 8$, that is $z^3 - 2z + 8 = 0$. B1 for the alternating signs, M1 for inserting the three values, A1 for the cubic. The relations run both ways, from roots to coefficients as directly as from coefficients to roots.

SECTION C · Stretch

C1 From the coefficients, $\Sigma\alpha = 4$, $\Sigma\alpha\beta = 1$ and $\alpha\beta\gamma = -6$. Squaring the first, $(\Sigma\alpha)^2 = \Sigma\alpha^2 + 2\Sigma\alpha\beta$, so $\Sigma\alpha^2 = 16 - 2 = \mathbf{14}$. For the cubes, use the equation itself. Each root satisfies $\alpha^3 = 4\alpha^2 - \alpha - 6$, so summing over the three roots gives $\Sigma\alpha^3 = 4\Sigma\alpha^2 - \Sigma\alpha - 18 = 56 - 4 - 18 = \mathbf{34}$. For the new cubic, its three symmetric functions are $\Sigma\alpha^2 = 14$, $\Sigma\alpha^2\beta^2 = (\Sigma\alpha\beta)^2 - 2(\Sigma\alpha)(\alpha\beta\gamma) = 1 + 48 = 49$, and $\alpha^2\beta^2\gamma^2 = (\alpha\beta\gamma)^2 = 36$. So the equation is $\mathbf{z^3 - 14z^2 + 49z - 36 = 0}$. B1 for the three symmetric functions, M1 for squaring the sum, A1 for 14, M1 for using the equation on each root, A1 for 34, M1 for $\Sigma\alpha^2\beta^2$, A1 for 49, B1 for the product 36, A1 for the cubic. The actual roots are 2, 3 and -1, whose squares 4, 9 and 1 do satisfy it, but the marks here are for the symmetric-function route. Squaring each coefficient of the original cubic is a common and completely invalid shortcut.
