inkMaths

ANSWER BOOK

Second order equations

Differential equations · Year FM

7 questions · 29 marks

NAVY EDITION

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SECTION A · Warm-up

A1 $am^2 + bm + c = 0$. Substituting the trial solution $y = e^{mx}$ puts a factor of e^{mx} in every term; cancelling it leaves the quadratic. B1 for the auxiliary equation, B1 for where it comes from. [2]

A2 Auxiliary: $m^2 - 7m + 12 = (m - 3)(m - 4) = 0$, roots 3 and 4. General solution $y = Ae^{3x} + Be^{4x}$: two distinct real roots, two independent exponentials. M1 for the auxiliary equation, A1 for the roots, A1 for the general solution. [3]

SECTION B · Standard

B1 $m^2 + 9 = 0$ gives $m = \pm 3i$, so $y = A \cos 3x + B \sin 3x$. Pure oscillation at angular frequency 3, so every solution repeats with period $2\pi/3$. M1 for $m = \pm 3i$, A1 for the general solution, B1 for the period. [3]

B2 $m^2 - 6m + 9 = (m - 3)^2 = 0$, a repeated root. The solution needs its extra factor of x, so $y = (A + Bx)e^{3x}$. M1 for the repeated root, A1 for the extra factor of x, A1 for the general solution. Without the Bx the two constants would collapse into one and two initial conditions could not be met. [3]

B3 $m^2 - 2m + 5 = 0$ gives $m = 1 \pm 2i$, so $y = e^x(A \cos 2x + B \sin 2x)$. $y(0) = 2$ gives $A = 2$. Differentiating and setting $x = 0$ gives $y'(0) = A + 2B = 0$, so $B = -1$. The solution is $y = e^x(2 \cos 2x - \sin 2x)$, an oscillation at frequency 2 growing inside e^x. M1 for the auxiliary equation, A1 for $m = 1 \pm 2i$, A1 for the general solution, B1 for $A = 2$, M1 for differentiating and using $y'(0) = 0$, A1 for $B = -1$. [6]

B4 $m^2 - 1 = 0$ gives $m = \pm 1$, so $y = Ae^x + Be^{-x}$. The conditions give $A + B = 1$ and $A - B = 0$, so $A = B = 1/2$ and $y = (e^x + e^{-x})/2 = \cosh x$. M1 for the auxiliary equation, A1 for the general solution, M1 for both conditions, A1 for A and B, B1 for naming $\cosh x$. The hyperbolic cosine is what this differential equation looks like when met without its name. [5]

SECTION C · Stretch

C1 The auxiliary equation $m^2 - 3m + 2 = 0$ has roots 1 and 2, so the complementary function is $Ae^x + Be^{2x}$. Trying $y = ke^{2x}$ for the particular integral fails, because that is already part of the complementary function and the left side collapses to zero. Multiply by x and try $y = kxe^{2x}$ instead. Then $y' = ke^{2x}(1 + 2x)$ and $y'' = ke^{2x}(4 + 4x)$, so the left side is $ke^{2x}[(4 + 4x) - 3(1 + 2x) + 2x] = ke^{2x}$. Matching to $4e^{2x}$ gives $k = 4$, so the general solution is $\mathbf{y} = Ae^x + Be^{2x} + 4xe^{2x}$. M1 for the auxiliary equation, A1 for the complementary function, B1 for spotting the duplication, M1 for trying kxe^{2x}, A1 for the substitution, A1 for $k = 4$, A1 for the general solution. Always check the trial function against the complementary function before differentiating. When the right-hand side duplicates a term of the complementary function, an extra factor of x is needed, and a repeated auxiliary root would require x^2. [7]

