InkMaths

ANSWER BOOK

Series solutions of differential equations

Further Pure 1 · Year FM

6 questions · 22 marks

NAVY EDITION

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SECTION A · Warm-up

A1 Rearrange for the highest derivative and substitute the initial values to find it at the starting point. Differentiate the whole equation and substitute again for the next derivative, and so on. Each derivative becomes one coefficient of the Taylor series of the solution. B1 for evaluating the highest derivative from the initial values, B1 for repeated differentiation supplying each further coefficient. [2]

A2 dy/dx at 0 is $0 + 1^2 = 1$. Differentiating the equation: the second derivative is $1 + 2y(dy/dx)$, which at 0 is $1 + 2(1)(1) = 3$. B1 for $dy/dx = 1$, M1 for differentiating the equation, A1 for 3. [3]

SECTION B · Standard

B1 The second derivative is $1 + 2y\,y'$. Differentiating that by the product rule gives $2(y')^2 + 2y\,y''$, which at the origin is $2(1)^2 + 2(1)(3) =$ **8**.
Taylor's series about 0 is $y = y(0) + y'(0)x + y''(0)x^2/2! + y'''(0)x^3/3!$, so $y = 1 + x + 3x^2/2! + 8x^3/3!$, that is $\mathbf{y} = 1 + x + 1.5x^2 + (4/3)x^3 + \dots$. M1 for differentiating by the product rule, A1 for 8, M1 for the Taylor form with its factorials, A1 for the series. Keep the factorials visible until the last line; dividing by n rather than n! is the standard slip. [4]

B2 $dz/dx = 2y(dy/dx)$, so the equation becomes $dz/dx + z = x$: linear in z. The integrating factor is e^x, giving $e^xz = \int x\,e^x\,dx = (x - 1)e^x + C$. So $z = x - 1 + Ce^{-x}$, and $y^2 = x - 1 + Ce^{-x}$. B1 for $dz/dx = 2y(dy/dx)$, M1 for the integrating factor e^x, A1 for $e^xz = (x - 1)e^x + C$, A1 for $y^2 = x - 1 + Ce^{-x}$. Substituting back into the original equation confirms it. [4]

B3 The equation delivers the second derivative in terms of x, y and dy/dx, so both y and dy/dx must be known at the starting point before anything can be evaluated. With only one condition the very first substitution is impossible. B1 B1 for the two points: the equation returns the second derivative in terms of y and dy/dx, and both must be known before any substitution. [2]

SECTION C · Stretch

C1 Rearrange for the highest derivative: $y'' = -xy' - 2y$. At the origin $y'' = -0 - 2(1) =$ **-2**. [7]

Differentiate, using the product rule on xy':
$y''' = -y' - xy'' - 2y' = \mathbf{-3y'} - xy''$, so $y'''(0) = \mathbf{0}$.
Again: $y'''' = -3y'' - y'' - xy''' = \mathbf{-4y''} - xy'''$, so $y''''(0) = -4(-2) = \mathbf{8}$.
The pattern holds, each differentiation raising the coefficient by one. The fifth derivative is $-5y''' - xy''''$, giving **0**, and the sixth is $-6y'''' - xy^{(5)}$, giving $-6(8) = \mathbf{-48}$.
Assembling Taylor's series:
$y = 1 + 0 - 2x^2/2! + 0 + 8x^4/4! + 0 - 48x^6/6!$
$= \mathbf{1} - x^2 + x^4/3 - x^6/15$.
Every odd derivative vanishes, because each is a multiple of the one two places before it and the chain starts at $y'(0) = 0$. Spotting that halves the work.
At $x = 0.2$: $y \approx 1 - 0.04 + 0.0016/3 - 0.000064/15 = 1 - 0.04 + 0.00053333 - 0.00000427 =$ **0.96053**.
M1 for rearranging for y'', A1 for $y''(0) = -2$, M1 for the repeated differentiation, A1 for $y''''(0) = 8$, A1 for -48 at the sixth derivative, A1 for the assembled series, B1 for 0.96053. The terms shrink by roughly a factor of a hundred each time, so the next one cannot disturb the fifth decimal place.