inkMaths

ANSWER BOOK

Subgroups, Lagrange's theorem and isomorphism

Further Pure 2 · Year FM

7 questions · 27 marks

NAVY EDITION

inkphysics.co.uk/maths/ · youtube.com/@InkPhysics

SECTION A · Warm-up

A1 The order of any subgroup of a finite group divides the order of the group. Since the powers of a single element form a subgroup, the order of every element divides the order of the group as well. B1 for the theorem, B1 for the order of an element dividing the order of the group. [2]

A2 The subgroups are $\{e\}$, $\{e, g^3\}$, $\{e, g^2, g^4\}$ and the whole group, of orders 1, 2, 3 and 6. Each of those divides 6, as Lagrange requires, and every divisor of 6 is achieved: that always happens in a cyclic group, though not in general. B1 for $\{e, g^3\}$, B1 for $\{e, g^2, g^4\}$ with the two trivial subgroups, B1 for the orders dividing 6. [3]

SECTION B · Standard

B1 Subgroup orders must divide 8, so they are 1, 2, 4 or 8. If some element a had order 3 then $\{e, a, a^2\}$ would be a subgroup of order 3, and 3 does not divide 8. That contradicts Lagrange, so no such element exists. B1 for the possible orders 1, 2, 4 and 8, M1 for constructing $\{e, a, a^2\}$, A1 for its order being 3, A1 for the contradiction with Lagrange. [4]

B2 By Lagrange any subgroup has order dividing p, so its order is 1 or p: the only subgroups are the identity and the whole group. Take any element a other than the identity. The subgroup generated by a is not $\{e\}$, so it must be the whole group, and therefore a generates G. Hence G is cyclic, and every non-identity element is a generator. M1 for applying Lagrange, A1 for the only subgroup orders being 1 and p, M1 for taking a non-identity element, A1 for concluding that G is cyclic. [4]

B3 They are not. The cyclic group of order 4 contains an element of order 4, its generator. In $\{1, 3, 5, 7\}$ every element squares to 1, so no element has order more than 2. An isomorphism preserves the order of every element, so no bijection between them can be one. B1 for an element of order 4 in the cyclic group, B1 for every element of $\{1, 3, 5, 7\}$ squaring to 1, B1 for isomorphism preserving orders. [3]

B4 The four rotations, through 0°, 90°, 180° and 270°, form a **cyclic** subgroup of order 4, generated by the quarter turn. [4]
The identity, the rotation through 180° and the two reflections in the lines through opposite edge midpoints form a **different** subgroup of order 4. It is closed, since the product of those two reflections is the half turn, and every element squares to the identity, so it has no element of order 4 and is not isomorphic to the first.
The identity with the rotation through 180° gives a subgroup of **order** 2. B1 for the rotation subgroup, M1 A1 for a second subgroup of order 4 with its closure, B1 for the subgroup of order 2. All three orders divide 8, as Lagrange requires.

SECTION C · Stretch

C1 Matching orders is not enough. An isomorphism preserves the order of every element, so compare how many elements of each order the two groups hold. [7]
In U. $7^2 = 49 \equiv 1$, $9^2 = 81 \equiv 1$ and $15^2 = 225 \equiv 1$, so 7, 9 and 15 have order 2. For 3: $3^2 = 9$, $3^3 = 27 \equiv 11$, $3^4 = 81 \equiv 1$, so 3 has order 4, and the same for 5, 11 and 13. The tally is **one** of order 1, three of order 2, four of order 4.
In D. The quarter turns, through 90° and 270°, have order 4. The half turn has order 2, and so does each of the four reflections, since reflecting twice in the same line does nothing. The tally is **one** of order 1, five of order 2, two of order 4.
The two lists differ, so **D** and U are not isomorphic.
A second argument settles it alone. U is a group of residues under multiplication, so it is **abelian**. D is not: rotating a square through 90° and then reflecting gives a different symmetry from reflecting first. An isomorphism carries a commuting pair to a commuting pair, so an abelian group cannot be isomorphic to a non-abelian one.
M1 for finding element orders in U, A1 for the tally in U, M1 for the element orders in D, A1 for the tally in D, A1 for the conclusion, B1 for U being abelian, B1 for D not being abelian.
Lagrange rules nothing out here. Both groups have all their element orders in $\{1, 2, 4\}$, every one of which divides 8. Lagrange says which orders are possible, never how many elements take each.

