inkMaths

ANSWER BOOK

Tangents, turning points and curve behaviour

Differentiation · Year 12–13

8 questions · 35 marks

NAVY EDITION

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SECTION A · Warm-up

A1 $dy/dx = 2x$, which is 4 at $x = 2$, so the tangent has gradient 4. Using $y - y_1 = m(x - x_1)$ gives $y - 4 = 4(x - 2)$, that is $y = 4x - 4$. M1 for differentiating, A1 for the gradient, A1 for the equation. The derivative supplies the slope and ordinary straight-line work does the rest. [3]

A2 The tangent gradient at $x = 2$ is $2x = 4$, and the normal is perpendicular to it, so its gradient is the negative reciprocal, $-¼$. Then $y - 4 = -¼(x - 2)$, which tidies to $y = -x/4 + 9/2$, or $x + 4y = 18$. M1 for the tangent gradient, M1 for the negative reciprocal, A1 for the equation. Check the perpendicularity by multiplying: $4 \times (-¼) = -1$. One derivative feeds both lines, and forgetting to invert as well as negate is the standard slip. [3]

SECTION B · Standard

B1 $dy/dx = 3x^2 - 6x - 9 = 3(x - 3)(x + 1)$, which is zero at $x = -1$ and $x = 3$. Substituting into the original equation gives $y(-1) = -1 - 3 + 9 + 5 = 10$ and $y(3) = 27 - 27 - 27 + 5 = -22$, so the stationary points are $(-1, 10)$ and $(3, -22)$. The second derivative is $6x - 6$, which is -12 at $x = -1$, a maximum, and $+12$ at $x = 3$, a minimum. M1 A1 for the derivative, M1 for solving, A1 for both x values, A1 for both y values, A1 for the classification. The heights come from the original curve, never from the derivative, and substituting into dy/dx to find y is a favourite way of losing two marks at the last moment. [6]

B2 A curve falls where its gradient is negative, so solve $dy/dx < 0$. Here $dy/dx = 3x^2 - 6x - 9 = 3(x - 3)(x + 1)$, an upward parabola in x, which is negative strictly between its roots. So C is decreasing for $-1 < x < 3$. M1 for setting the derivative below zero, M1 for factorising or solving, A1 for the interval. Between the hilltop at $x = -1$ and the valley at $x = 3$ the only way is down. [3]

B3 $dy/dx = 3x^2$, which is zero at $x = 0$, so the origin is a stationary point. The second derivative $6x$ is also zero there, so that test decides nothing. Instead look at the sign of dy/dx on either side: $3x^2$ is positive for $x = -1$ and positive for $x = 1$, so the curve climbs into the point and climbs away from it. It is therefore a point of inflection with a horizontal tangent. M1 for the stationary point, M1 for testing the gradient either side, A1 for the conclusion. When $f'' = 0$ the test is silent rather than negative, and the gradient's behaviour on both sides is what settles the classification. [3]

B4 If one side is x then the other is $20 - x$, so $A = x(20 - x) = 20x - x^2$. Then $dA/dx = 20 - 2x$, which is zero at $x = 10$, and $d^2A/dx^2 = -2$, negative, confirming a maximum. The rectangle is a 10 cm by 10 cm square with area 100 cm^2. M1 for the second side in terms of x, A1 for the area function, M1 for differentiating and solving, A1 for $x = 10$, A1 for the area with its justification. Write the quantity as a function of one variable first, then let the calculus do the choosing. Skipping the justification loses a mark even when the answer is right. [5]

SECTION C · Stretch

C1 With height h, the volume gives $\pi r^2 h = 1000$, so $h = 1000/(\pi r^2)$. The closed cylinder has two circular ends and a curved surface, so $S = 2\pi r^2 + 2\pi rh = 2\pi r^2 + 2\pi r \times 1000/(\pi r^2) = 2\pi r^2 + 2000/r$, as required. Differentiating, $dS/dr = 4\pi r - 2000/r^2$, and setting this to zero gives $4\pi r^3 = 2000$, so $r^3 = 500/\pi$ and $r = 5.42$ cm. For the justification, $d^2S/dr^2 = 4\pi + 4000/r^3$, which is positive for every $r > 0$, so the stationary point is a minimum. M1 for h from the volume, M1 for substituting into S, A1 for the printed result, M1 for differentiating, A1 for dS/dr, A1 for $r = 5.42$, B1 for the second derivative justification. Writing $2000/r$ as $2000r^{-1}$ before differentiating avoids the most common error, which is differentiating the fraction to $-2000/r$ and losing the power altogether. The minimum area is about 554 cm^2. [7]

C2 Write $y = x + 4x^{-1}$, so $dy/dx = 1 - 4x^{-2} = 1 - 4/x^2$. Setting this to zero gives $x^2 = 4$, so $x = 2$, taking the positive root because $x > 0$. Then $y = 2 + 2 = 4$. For the justification, $d^2y/dx^2 = 8x^{-3} = 8/x^3$, which is 1 at $x = 2$ and therefore positive, confirming a minimum. M1 for rewriting with a negative index, A1 for the derivative, M1 for solving, A1 for $x = 2$, A1 for $y = 4$ with justification. The domain restriction is doing real work here, since $x = -2$ is also stationary and gives a local maximum of -4, which sits below the minimum you have just found. Local extrema are local, and comparing values across different branches of a curve is meaningless. [5]