inkMaths

ANSWER BOOK

The Central Limit Theorem

Further Statistics 1 · Year FM

7 questions · 27 marks

NAVY EDITION

inkphysics.co.uk/maths/ · youtube.com/@InkPhysics

SECTION A · Warm-up

A1 For a population with mean μ and variance σ^2, the mean of a random sample of size n is approximately $N(\mu, \sigma^2/n)$ when n is large, whatever the shape of the parent distribution. The approximation improves as n grows. B1 for the mean, B1 for the variance σ^2/n, B1 for it holding whatever the shape of the parent. State that the sample is random and that n is large, since both conditions carry credit. [3]

A2 Approximately **N(75,** 4), since 256/64 = 4. The standard error is $\sqrt{4}$ = **2**, which is also $16/\sqrt{64}$. M1 for 256/64, A1 for the distribution, B1 for the standard error 2. Divide the variance by n rather than the standard deviation, and write the distribution with a variance in the bracket, not a standard deviation. [3]

SECTION B · Standard

B1 $z = (78 - 75)/2 =$ **1.5**, so $P(\bar{X} > 78) = P(Z > 1.5) = 1 - 0.9332 =$ **0.0668**. M1 for standardising, A1 for z = 1.5, M1 for 1 - 0.9332, A1 for 0.0668. Dividing by the standard error of the mean rather than by 16 is the whole point, and the parent's shape never entered the working. [4]

B2 A Poisson has mean and variance both 9, so the sample mean is approximately $N(9, 9/36) = N(9, 0.25)$, standard error 0.5. Then $z = (8.5 - 9)/0.5 = -1$, giving $P \approx 0.1587$. B1 for the mean and variance both being 9, M1 for N(9, 0.25), M1 for z = -1, A1 for 0.1587. A discrete, skewed parent is no obstacle at all. [4]

B3 $6/\sqrt{n} \le 0.5$ gives $\sqrt{n} \ge 12$, so $n \ge 144$. M1 for $6/\sqrt{n} \le 0.5$, A1 for $\sqrt{n} \ge 12$, A1 for n = 144. Halving that requirement to 0.25 would need $n \ge 576$: four times the data for half the error, because the root is what does the work. [3]

B4 Individual observations keep the parent's distribution no matter how many are collected, since taking more of them changes nothing about any single one. What becomes approximately normal is the distribution of the sample mean, taken over repeated samples of size n. The theorem is a statement about a statistic, not about the raw data. B1 for individual observations keeping the parent distribution, B1 for the sample mean being what becomes normal, B1 for the theorem being a statement about a statistic. [3]

SECTION C · Stretch

C1 By the Central Limit Theorem $\bar{X}$ is approximately $N(\mu, 25/n)$ for large n, so $Z = (\bar{X} - \mu)/(5/\sqrt{n})$ is approximately standard normal. [7]
$P(|\bar{X} - \mu| < 1) \geq 0.95$ requires $1 \geq 1.96 \times 5/\sqrt{n}$, so $\sqrt{n} \geq 9.8$ and $n \geq 96.04$. The normal approximation therefore estimates the least such n as **n** = 97.
Check the boundary: with n = 96 the margin is $1.96 \times 5/\sqrt{96} = 1.0002$, just too wide, and with n = 97 it is 0.9950. Always round a sample size up, whatever the decimal, since rounding down breaks the requirement.
Now say what the theorem does and does not give you. It makes $\bar{X}$ approximately normal for large n without any assumption about the parent, which is why no shape was ever named, but it is an approximation rather than a guarantee at any fixed n. How large n must be before that approximation can be trusted depends on the parent, since a heavily skewed or long-tailed population settles down more slowly. So 97 is the least n the normal approximation estimates, not a smallest n that is certain to work for every population. An exact minimum would need the parent distribution itself, or a result that bounds the error in the normal approximation. The variance also has to be known, and here it is 25. Halving the margin to 0.5 would need $n \geq 385$, roughly four times as many observations. M1 for $N(\mu, 25/n)$, A1 for the standardised form, M1 for $1 \geq 1.96 \times 5/\sqrt{n}$, A1 for $\sqrt{n} \geq 9.8$, A1 for n = 97, B1 for the theorem needing no assumption about the parent, B1 for it being an approximation rather than a guarantee.