inkMaths

ANSWER BOOK

# The derivative from first principles

Differentiation · Year 12–13

6 questions · 22 marks

NAVY EDITION


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## SECTION A · Warm-up

**A1** $f'(x) = \lim$ as $h \to 0$ of $[f(x + h) - f(x)]/h$. The fraction is the gradient of the chord joining $x$ to $x + h$, and the limit slides the far end of that chord into the near end, turning a chord into a tangent. B1 for the definition, B1 for the chord becoming a tangent. Writing the quotient without the limit sign scores nothing, since the limit is the definition. [2]

**A2** With $h = 0.1$ the chord gradient is $(3.1^2 - 9)/0.1 = 6.1$. With $h = 0.01$ it is $6.01$. The values are settling towards 6, which the derivative $2x$ confirms at $x = 3$. M1 for a chord gradient, A1 for 6.1, A1 for 6.01. Numerical chords are evidence, never proof, so a question asking you to prove the result still needs the algebra. [3]

## SECTION B · Standard

**B1** The chord gradient is $[(x + h)^2 - x^2]/h = (2xh + h^2)/h$. For $h \neq 0$ this cancels to $2x + h$, and as $h \to 0$ what survives is $2x$. Three moves earn the marks: expand, divide, let $h$ tend to zero. M1 for the chord gradient, A1 for the expansion, A1 for cancelling to $2x + h$, A1 for the limit. The middle move requires $h \neq 0$ to be stated, and the words as $h \to 0$ are the ones doing the work, rather than a bare substitution of $h = 0$. [4]

**B2** The chord gradient is $[3(x + h)^2 + 2(x + h) - 3x^2 - 2x]/h = (6xh + 3h^2 + 2h)/h$, which is $6x + 3h + 2$ for $h \neq 0$. As $h \to 0$ it becomes $6x + 2$. M1 for the chord gradient, A1 for $6xh + 3h^2 + 2h$, A1 for $6x + 3h + 2$, A1 for $6x + 2$. Every term of f contributes its own derivative, and that is the termwise rule being born. Forgetting to expand $3(x + h)^2$ fully is the usual source of a lost $6xh$. [4]

**B3** The chord gradient from $x = 2$ is $[(2 + h)^3 - 8]/h = (12h + 6h^2 + h^3)/h = 12 + 6h + h^2$. As $h \to 0$ the gradient is 12. M1 for the chord gradient from $x = 2$, A1 for the expansion, A1 for $12 + 6h + h^2$, A1 for 12. Working at the specific point keeps the binomial expansion short, since the 8 cancels and only the h terms remain. [4]

## SECTION C · Stretch

**C1** The chord gradient is $[1/(x + h) - 1/x]/h$. Combine the two fractions over the common denominator $x(x + h)$, giving $[x - (x + h)]/[x(x + h)h] = -h/[x(x + h)h]$. For $h \neq 0$ the h cancels to leave $-1/[x(x + h)]$, and as $h \to 0$ that tends to $-1/x^2$. M1 for the chord gradient, M1 for combining over $x(x + h)$, A1 for $-h/[x(x + h)h]$, A1 for cancelling, A1 for the limit. The difficulty is entirely algebraic. The subtraction has to happen inside one fraction before the division by h can cancel anything, and candidates who write $1/(x + h) - 1/x = 1/h$ lose every mark. Note also that the h in the denominator only cancels because $h \neq 0$ throughout the working. [5]