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ANSWER BOOK

The Newton–Raphson method

Numerical methods · Year 13

15 questions · 57 marks

NAVY EDITION

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SECTION A · Warm-up

A1 NRT-A1

$x_{n+1} = x_n - f(x_n)/f'(x_n)$. B1 for the formula, B1 for the tangent interpretation. [2]
Geometrically it slides down the tangent at the current point to where that tangent crosses the axis, then starts again from there. The minus sign is part of the formula, not an optional extra.

A2 NRT-A2

On the x-axis $y = 0$, so the equation of the tangent becomes $-f(x_n) = f'(x_n)(x - x_n)$. M1 for setting $y = 0$ in the equation of the tangent. Dividing by $f'(x_n)$ gives $x - x_n = -f(x_n)/f'(x_n)$, so $x = x_n - f(x_n)/f'(x_n)$, which is the printed result. A1 for the rearrangement. That crossing is what the method takes as the next estimate, so one Newton-Raphson step is one tangent drawn and followed to the axis, and the formula is a record of that construction rather than a rule to be memorised on its own. The division requires $f'(x_n)$ to be non-zero, which is the case a horizontal tangent excludes. [2]

A3 NRT-A3

$f'(x) = 2x$, so the iteration reads $x_{n+1} = x_n - (x_n^2 - 5)/(2x_n)$. With $x_0 = 2$: $f(2) = -1$ and $f'(2) = 4$, so $x_1 = 2 - (-1)/4 = 9/4$. M1 for the substitution into the formula, A1 for 9/4. Two minus signs meet in that step, so the estimate moves up towards $\sqrt{5} = 2.2360\ldots$ rather than down. [2]

A4 NRT-A4

The step takes the tangent to $y = f(x)$ at the point $(x_n, f(x_n))$ and reads x_{n+1} off where that tangent meets the x-axis. B1 for the tangent at the current point, B1 for the crossing of the x-axis. A small gradient makes a shallow tangent, which meets the axis a long way from x_n, so the next iterate can land far from the root asked for and later steps may settle on a different root instead. B1 for the long step and what it can lead to. Where $f'(x_n)$ is exactly zero the tangent is horizontal and no next iterate exists at all. [3]

SECTION B · Standard

B1 NRT-B1

$f(1.5) = -0.125$ and $f'(1.5) = 3(1.5)^2 + 1 = 7.75$, so $x_1 = 1.5 + 0.125/7.75 = 1.516129\ldots$. [4]
Then $f(x_1) = 0.00117\ldots$, $f'(x_1) = 7.8959\ldots$, and $x_2 = 1.51598$ to 5 decimal places. M1 for f and f′ at 1.5, A1 for the first iterate, M1 for the second iteration, A1 for 1.51598. The root is $1.5159802\ldots$, so x_1 was right to 3 decimal places and x_2 is right to 7. Near a simple root, and once the iterates are sufficiently close under the usual smoothness conditions, Newton-Raphson normally has quadratic convergence. Keep unrounded intermediate values: rounding x_1 to 1.5161 before the second step corrupts the fifth decimal place of x_2.

B2 NRT-B2 [4]

$f(2) = \ln 2 - 1 = -0.30685 < 0$ and $f(3) = \ln 3 = 1.09861 > 0$, and f is continuous for $x > 0$, so a root of $f(x) = 0$ lies in $(2, 3)$. B1 for the two values with the sign change and the continuity. $f'(x) = 1/x + 1$, so $f'(2) = 1.5$ and $x_1 = 2 + 0.30685/1.5 = 2.20457$. Then $f(x_1) = -0.004900$ and $f'(x_1) = 1.45360$, so $x_2 = 2.2079$ to 4 decimal places. M1 for f and f′ at $x_0 = 2$, A1 for x_1, A1 for 2.2079. The answer lies inside $(2, 3)$, which is where the sign change put the root, so the iteration has converged to the root the first part located. Locating the root before iterating is what supplies the starting value here, and testing the ends of the rounding interval settles the last figure: $f(2.20785) = -0.000131$ and $f(2.20795) = 0.000014$.

B3 NRT-B3 [2]

$f'(x) = 3x^2 - 2$. At $x_0 = 0$, $f(0) = 2$ and $f'(0) = -2$, so $x_1 = 0 - 2/(-2) = 1$. At $x_1 = 1$, $f(1) = 1$ and $f'(1) = 1$, so $x_2 = 1 - 1/1 = 0$. M1 for both steps. The iterates repeat 0, 1, 0, 1, … without end, so the sequence never approaches the only root of $f(x) = 0$, which is at $x = -1.769$ to 3 decimal places: the method does not converge from $x_0 = 0$. A1 for the repeating pair and that conclusion. Neither $f'(0)$ nor $f'(1)$ is zero, so both steps are defined and the arithmetic is correct throughout; a starting value can be unusable without any tangent being horizontal. $f(-2) = -2$ and $f(-1) = 3$, so a sign change locates the root in $(-2, -1)$ and a start there reaches it.

B4 NRT-B4 [4]

$f'(x) = e^x - 3$. At $x_0 = 1.5$: $f(1.5) = 4.4817 - 4.5 = -0.018311$ and $f'(1.5) = 1.481689$, so $x_1 = 1.5 + 0.018311/1.481689 = 1.512358$. Then $f(x_1) = 0.00034\ldots$ and $f'(x_1) = 1.5374\ldots$, so $x_2 = 1.51213$ to 5 decimal places. M1 for f and f′ at 1.5, A1 for x_1, M1 for the second iteration, A1 for 1.51213. The value lies between 1.5 and 2, which is where the root was located, so the iteration has converged to the one asked for. Keep x_1 unrounded on the calculator: entering 1.5124 for the second step moves the fifth decimal place of x_2.

B5 NRT-B5 [4]

$f'(x) = 3x^2 - 12x + 9 = 3(x - 1)(x - 3)$, so $f'(3) = 0$, while $f(3) = 27 - 54 + 27 - 1 = -1$ is not zero. The tangent at $x = 3$ is horizontal, so it never meets the x-axis and the formula divides by zero: no first iterate exists there. B1 for $f'(3) = 0$, B1 for $f(3)$ not being zero and the tangent being horizontal. From $x_0 = 3.2$: $f(3.2) = -0.872$ and $f'(3.2) = 1.32$, so $x_1 = 3.2 + 0.872/1.32 = 3.8606$ to 4 decimal places. M1 for the iteration, A1 for 3.8606. The derivative is still small at 3.2, so the step is long: it travels 0.66 and passes the root at 3.5321, which later steps come back to.

B6 NRT-B6 [4]

Setting $s = 100$ gives $50t + 250(e^{-0.2t} - 1) = 100$, and t cannot be made the subject because it appears both inside and outside the exponential. So take $f(t) = 50t + 250(e^{-0.2t} - 1) - 100$ and find a root. B1 for forming the equation. $f'(t) = 50 - 50e^{-0.2t}$, and at $t_0 = 5$: $f(5) = -8.0301$ and $f'(5) = 31.6060$, so $t_1 = 5 + 8.0301/31.6060 = 5.25407$, and a second step gives $t_2 = 5.25048$. M1 for f′, M1 for the iteration, A1 for 5.25 seconds. The value lies between 5 and 6, as the question said it would. Give the time in seconds to the 3 significant figures asked for: 5.25048 s claims a precision the model does not carry.

B7 NRT-B7 [5]

$f'(x) = 3x^2 - 2$. At $x_0 = 2$: $f(2) = -1$ and $f'(2) = 10$, so $x_1 = 2 + 1/10 = 2.1$ exactly. Then $f(2.1) = 0.061$ and $f'(2.1) = 11.23$, so $x_2 = 2.1 - 0.061/11.23 = 2.094568$ to 6 decimal places. M1 for f and f′ at 2, A1 for $x_1 = 2.1$, M1 for the second iteration, A1 for 2.094568. The distances from α are 0.0946, 0.0054 and 0.0000166, each one of the order of the square of the one before it, which is the quadratic convergence expected near a simple root from a suitable starting value. B1 for the comparison. Two steps have bought about four more correct decimal places than the starting value had.

SECTION C · Stretch

C1 NRT-C1 [6]

$f'(x) = 3x^2 - 3$, so $f'(1) = 0$ while $f(1) = -1$. The Newton-Raphson formula would divide by zero, so no next iterate is defined from $x_0 = 1$. The tangent there is horizontal. For $x_0 = 0.9$, $f(0.9) = -0.971$ and $f'(0.9) = -0.57$, giving $x_1 = 0.9 - (-0.971)/(-0.57) = -0.8035\ldots \approx -0.804$. The small derivative produces a large step. Continuing from this value converges to the root near 1.532, not the nearer root near 0.347. B1 for $f'(1) = 0$, B1 for the division being undefined, B1 for the horizontal tangent, M1 for the iteration from 0.9, A1 for the value of x_1, B1 for the comment on the root reached. A sign-change interval or sketch should be used to identify the intended root before iteration.

C2 NRT-C2 [3]

$f'(x) = 1/x$, so $x_{n+1} = x_n - \ln x_n/(1/x_n) = x_n - x_n \ln x_n = x_n(1 - \ln x_n)$. M1 for the substitution into the formula, A1 for the printed form. From $x_0 = 3$: $x_1 = 3(1 - \ln 3) = 3(1 - 1.09861) = -0.29584$. $\ln x$ is defined only for $x > 0$, so $f(x_1)$ cannot be evaluated and no x_2 exists: the method breaks down at the second step rather than the first. B1 for the negative first iterate with that reason. The gradient at $x = 3$ is only $1/3$, so the tangent there is shallow and meets the axis 3.29584 to the left of x_0, on the far side of the vertical asymptote of $y = \ln x$ at $x = 0$. From $x_0 = 1.5$ the same iteration gives 0.89180, 0.99392 and 0.99998, converging to the root at $x = 1$, so what fails here is the starting value and not the method.

C3 NRT-C3 [6]

With $f(x) = x^2 - a$ and $f'(x) = 2x$, the formula gives $x_{n+1} = x_n - (x_n^2 - a)/(2x_n) = (x_n^2 + a)/(2x_n)$, which is $\frac{1}{2}(x_n + a/x_n)$. M1 for the substitution, A1 for the printed form. With $a = 10$ from $x_0 = 3$: $x_1 = \frac{1}{2}(3 + 10/3) = 3.166667$, $x_2 = 3.162281$ and $x_3 = 3.162278$ to 6 decimal places, against $\sqrt{10} = 3.1622776\ldots$. M1 for a correct iterate, A1 for x_3. For the identity, subtract $\sqrt{a}$ from the iteration and put the result over the common denominator: $x_{n+1} - \sqrt{a} = (x_n^2 + a - 2x_n\sqrt{a})/(2x_n)$, and the numerator is the square $(x_n - \sqrt{a})^2$, which is the printed result. M1 for the common denominator, A1 for recognising the square. The right-hand side is positive for every positive x_n, so every iterate after the first sits above $\sqrt{a}$, and the error at each step is the previous error squared and then divided by about $2\sqrt{a}$.

C4 NRT-C4 [6]

Setting $V = 10$ gives $\pi h^2(6 - h)/3 = 10$, so take $f(h) = \pi h^2(6 - h)/3 - 10$ and find a root. B1 for forming the equation. Differentiating, $f'(h) = \pi(12h - 3h^2)/3 = \pi(4h - h^2)$. M1 for f'. At $h_0 = 1.5$: $f(1.5) = 0.60288$ and $f'(1.5) = 11.7810$, so $h_1 = 1.5 - 0.60288/11.7810 = 1.44883$, and a second step gives $h_2 = 1.44846$. M1 for the iteration, A1 for h_1, A1 for the depth 1.45 m. The cubic has two more roots, near $h = -1.155$ and $h = 5.707$, and neither describes the tank: a depth is not negative, and it cannot exceed the diameter of 4 m. B1 for rejecting both, with a reason. Reading the model before iterating is what fixes the starting value: 1.5 sits inside $0 \le h \le 4$ and the other roots are outside it.

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