inkMaths

ANSWER BOOK

The product, quotient and chain rules

Differentiation · Year 12–13

9 questions · 30 marks

NAVY EDITION

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SECTION A · Warm-up

A1 Differentiate the outside first, leaving the inside untouched, to get $4(2x + 1)^3$, then multiply by the inside's derivative, 2. So $dy/dx = 8(2x + 1)^3$. M1 A1. Rates through a chain multiply, and the missing factor of 2 is the single most common omission in the topic. [2]

A2 The outside is e to something, which is its own derivative, and the inside x^2 contributes 2x. So $dy/dx = 2x\,e^{x^2}$. M1 A1. The exponential never changes shape, so the chain factor is doing all the varying. [2]

SECTION B · Standard

B1 With $u = x^2 - 3$, $dy/du = 6u^5$ and $du/dx = 2x$, so $dy/dx = 12x(x^2 - 3)^5$. Setting this to zero gives $x = 0$ or $x^2 = 3$, so $x = 0$, $\sqrt{3}$ and $-\sqrt{3}$. The corresponding y values are $(-3)^6 = 729$ at $x = 0$ and 0 at both $x = \pm\sqrt{3}$, so the stationary points are $(0, 729)$, $(\sqrt{3}, 0)$ and $(-\sqrt{3}, 0)$. M1 for the chain rule, A1 for the derivative, M1 for solving, A1 for all three x values, A1 for the points. Setting only the bracket to zero misses $x = 0$ entirely, and a product is zero when either factor is. [5]

B2 Product rule with $u = x^3$ and $v = e^{2x}$: $dy/dx = 3x^2e^{2x} + 2x^3e^{2x}$, which factorises as $x^2e^{2x}(3 + 2x)$. M1 for the rule, A1 for the two terms, A1 for the factorised form. Each factor takes a turn being differentiated while the other one waits, and the chain rule is still needed inside the second term to produce the 2. [3]

B3 Product rule: $dy/dx = 1 \times \sin x + x \times \cos x = \sin x + x \cos x$. At $x = \pi$, $\sin \pi = 0$ and $\cos \pi = -1$, so the gradient is $0 + \pi(-1) = -\pi$. M1 for the rule, A1 for the derivative, A1 for $-\pi$. Differentiating each factor and multiplying the results would give $\cos x$ on its own, and that is wrong. The rule shares the differentiation out one factor at a time. [3]

B4 Quotient rule with $u = 2x + 1$ and $v = x^2 + 1$: $dy/dx = [2(x^2 + 1) - (2x + 1)(2x)]/(x^2 + 1)^2$, which simplifies to $(-2x^2 - 2x + 2)/(x^2 + 1)^2$. At $x = 0$ the gradient is $2/1 = 2$. M1 for the rule, A1 for the numerator, A1 for the simplification, A1 for the value. The minus sign makes the order $u'v - uv'$ unforgiving, and swapping the two products flips the sign of every answer you will ever get from it. [4]

B5 Read the expression as $(\sin x)^3$. The outside gives $3(\sin x)^2$ and the inside gives $\cos x$, so $dy/dx = 3 \sin^2 x \cos x$. M1 A1. The notation $\sin^3 x$ hides the chain, and rewriting it with a bracket makes both layers visible before you start. [2]

SECTION C · Stretch

C1 Product rule with $u = x^2$ and $v = \ln 3x$. Note that $\ln 3x = \ln 3 + \ln x$, so its derivative is $1/x$. Then $dy/dx = 2x \ln 3x + x^2 \times 1/x = 2x \ln 3x + x = x(2 \ln 3x + 1)$, as required. At a stationary point, $x > 0$ so the bracket must vanish: $2 \ln 3x + 1 = 0$ gives $\ln 3x = -½$, so $3x = e^{-½}$ and $x = 1/(3\sqrt{e})$. Then $y = x^2 \ln 3x = (1/(9e)) \times (-½) = -1/(18e)$. The stationary point is $(1/(3\sqrt{e}), -1/(18e))$. M1 for the product rule, A1 for each term, A1 for the printed form, M1 for solving the bracket, A1 for x, A1 for y. The derivative of $\ln 3x$ is $1/x$ rather than $1/(3x)$. The chain rule gives $3/(3x)$, and the threes cancel. [6]

C2 With $u = \sin x$ and $v = \cos x$, the quotient rule gives $[\cos x \times \cos x - \sin x \times (-\sin x)]/\cos^2 x = (\cos^2 x + \sin^2 x)/\cos^2 x$. The numerator is 1 by the Pythagorean identity, so the derivative is $1/\cos^2 x = \sec^2 x$. M1 for the quotient rule, A1 for the collected numerator, A1 for the conclusion. The double negative in $-\sin x \times (-\sin x)$ is what makes the numerator add rather than cancel, and losing it produces $(\cos^2 x - \sin^2 x)/\cos^2 x$, which is $\cos 2x \sec^2 x$ and goes nowhere. [3]