inkMaths

ANSWER BOOK

The structure of proof: deduction and exhaustion

Proof · Year 12–13

8 questions · 26 marks

NAVY EDITION

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SECTION A · Warm-up

A1 An even number is $2n$ and an odd number is $2n + 1$, where n is an integer. Consecutive odd numbers are $2n + 1$ and $2n + 3$. B1 for $2n$, B1 for $2n + 1$, B1 for the consecutive odd pair. Declaring that n is an integer is part of the answer, because the letter is there precisely to stand for every integer at once. [3]

A2 Fifty checks are evidence, and evidence is not proof: case 51 could still fail. A proof must be a general argument that covers every case at once, by deduction from known facts, or by exhaustion when the cases form a complete finite list. B1 for why fifty checks are not a proof, B1 for naming a general deductive argument. [2]

SECTION B · Standard

B1 Complete the square, giving $n^2 + 4n + 7 = (n + 2)^2 + 3$. A square is never negative, so $(n + 2)^2 \geq 0$ and therefore $(n + 2)^2 + 3 \geq 3 > 0$. M1 for completing the square, A1 for $(n + 2)^2 + 3$, B1 for the non-negative square. Completing the square rewrites the expression so its sign becomes visible; that is the whole method. The sentence naming the square as non-negative carries a mark of its own, so the algebra alone is not the proof. [3]

B2 Let the numbers be $2n + 1$ and $2n + 3$ for an integer n. Their sum is $4n + 4 = 4(n + 1)$, and $n + 1$ is an integer, so the sum is 4 times an integer. B1 for $2n + 1$ and $2n + 3$, M1 for forming the sum, A1 for $4(n + 1)$ with the conclusion. The final sentence matters: exhibiting the factor of 4 and saying the other factor is an integer is what "divisible by 4" means. [3]

B3 Take the numbers as $2n + 1$ and $2n + 3$ for an integer n. Then $(2n + 3)^2 - (2n + 1)^2 = (4n^2 + 12n + 9) - (4n^2 + 4n + 1) = 8n + 8 = 8(n + 1)$, which is 8 times an integer. B1 for the two forms, M1 for expanding and subtracting, A1 for $8n + 8$, A1 for the conclusion. Expanding both squares fully before subtracting keeps the sign errors out, and the difference of two squares gives the same result in one line: $(4n + 4)(2) = 8(n + 1)$. Testing 3 and 5 proves nothing on its own. [4]

B4 Every integer is even or odd, so two cases cover everything. If $k = 2n$ then $k^2 = 4n^2$, a multiple of 4. If $k = 2n + 1$ then $k^2 = 4n^2 + 4n + 1 = 4(n^2 + n) + 1$, one more than a multiple of 4. B1 for splitting into even and odd, M1 for squaring each case, A1 for the even case, A1 for the odd case. This is proof by exhaustion over types rather than individual numbers: the two cases are exhaustive because every integer is one or the other. [4]

SECTION C · Stretch

C1 The complete list is 7, 11, 13, 17, 19. In turn, $p^2 - 1$ gives 48, 120, 168, 288, 360, and dividing by 24 gives 2, 5, 7, 12, 15, all integers. Every case on the complete list has been checked, so the claim is proved. B1 for the complete list, M1 for computing $p^2 - 1$, A1 for the five values, A1 for dividing by 24, B1 for the concluding statement. Listing the primes correctly is the first mark: miss one and the proof collapses. [5]

C2 Exhaustion needs a complete finite list of cases, and the integers go on for ever: however many are checked, infinitely many remain. B1 for exhaustion needing a finite list, B1 for the integers being infinite. A claim about all integers needs a general argument, so the deduction proof via the completed square is the only route. [2]