inkMaths

ANSWER BOOK

The trapezium rule

Numerical methods · Year 13

16 questions · 68 marks

NAVY EDITION

Every question in this book has a code beside its number.
Type inklearning.co.uk/q/ followed by the code to open that question online.

inklearning.co.uk/maths/ · youtube.com/@InkPhysics

SECTION A · Warm-up

A1 TRA-A1 [2]

Seven ordinates bound six strips, since n strips need n + 1 ordinates: the first and last ordinates are the ends of the interval, and each interior ordinate is shared by two neighbouring strips. B1 for 6 strips. Then $h = (3 - 0)/6 = 0.5$. B1 for $h = 0.5$. Dividing by 7 instead gives $h = 0.4286$ and a set of ordinates that never reaches $x = 3$, which is the error the ordinate count guards against.

A2 TRA-A2 [3]

$h = 2/4 = 0.5$, and the five ordinates at $x = 0, 0.5, 1, 1.5, 2$ are 1, $\sqrt{1.125} = 1.06066$, $\sqrt{2} = 1.41421$, $\sqrt{4.375} = 2.09165$ and 3. So $T = (0.5/2)[1 + 3 + 2(1.06066 + 1.41421 + 2.09165)] = 0.25 \times 13.13304 =$ **3.2833**. M1 for the five ordinates with $h = 0.5$, M1 for the bracket with the end ordinates once and the interior ones twice, A1 for 3.2833. This integrand has no antiderivative among the standard functions, which is the situation the rule exists for. Keep five figures in the ordinates and round once, at the end.

A3 TRA-A3 [2]

$L = h(y_0 + y_1 + \dots + y_{n-1})$ and $R = h(y_1 + \dots + y_n)$: the same ordinates and the same h, with the last ordinate left out of L and the first left out of R. B1 for both sums. They bound the integral when f is monotonic on every strip. Increasing throughout puts L below the integral and R above it, and decreasing throughout swaps them, because the argument uses the two end ordinates of each strip as its least and greatest values. B1 for the monotonic condition with the direction. A function that turns inside a strip takes values beyond both end ordinates there, and neither sum is then a bound.

A4 TRA-A4 [3]

Distance is the area under speed against time. The five readings are the ordinates, four strips, $h = 10$ s, so $T = (10/2)[0 + 15.6 + 2(6.4 + 11.3 + 14.1)] = 5 \times 79.2 = 396$. M1 for the bracket with the readings as ordinates and $h = 10$, A1 for 396. Metres per second multiplied by seconds gives metres, so **396** m, or about 400 m to the precision five readings support. The estimate assumes the speed changes steadily between readings, that is, each pair of consecutive readings is joined by a straight line. B1 for the units and the assumption. Nothing was measured inside any 10-second gap, so a surge or a hesitation within one leaves no trace in the arithmetic.

SECTION B · Standard

B1 TRA-B1 [4]

$h = 0.25$, and the ordinates at $x = 0, 0.25, 0.5, 0.75, 1$ are 1, $e^{0.0625} = 1.06449$, $e^{0.25} = 1.28403$, $e^{0.5625} = 1.75505$ and $e = 2.71828$. $T = (0.25/2)[1 + 2.71828 + 2(1.06449 + 1.28403 + 1.75505)] = 0.125 \times 11.92542 =$ **1.4907**. M1 for the ordinates, A1 for 1.4907. Differentiating twice, $f'(x) = 2xe^{x^2}$ and $f''(x) = (2 + 4x^2)e^{x^2}$, which is positive for every x, so on $[0, 1]$ the curve is convex, every chord lies above it and the estimate is an **overestimate**. M1 for f'', A1 for convex and therefore too big, with the chord above the curve as the reason. The true value is 1.4627, so the estimate is 0.028 high. The direction comes from the sign of f'' across every strip, not from its value at a single point.

B2 TRA-B2

Halve the strip width, that is, double the number of strips: shorter chords lie closer to the arcs they replace. B1 for more strips. Eight strips need nine ordinates, at $x = 0, 0.25, \ldots, 2$ with $h = 0.25$: 1, 1.00778, 1.06066, 1.19242, 1.41421, 1.71847, 2.09165, 2.52178 and 3. Every other one is an ordinate already used; the four between them are new. $T = (0.25/2)[1 + 3 + 2(1.00778 + 1.06066 + 1.19242 + 1.41421 + 1.71847 + 2.09165 + 2.52178)] = 0.125 \times 26.01394 =$ **3.2517**. M1 for the nine ordinates with $h = 0.25$, A1 for 3.2517. The error has fallen from 0.0420 to 0.0104, about a quarter of what it was, and it has kept its sign: $f''(x) = (3x + \frac{3}{4}x^4)(1 + x^3)^{-3/2}$ is positive on $[0, 2]$, so every estimate is an overestimate and the refined value approaches the true one from above. B1 for the error being smaller and of the same sign. Halving h roughly quarters the trapezium rule's error, which is what makes the extra ordinates worth computing. [4]

B3 TRA-B3

The list stops one ordinate short: the sixth strip runs from 2.5 to 3, and its right-hand side, $f(3)$, is missing. B1 for the missing ordinate at $x = 3$. Six strips have seven ordinates, since the ends of the interval count once and each of the five interior points is shared by two strips; the bracket takes $f(0)$ and $f(3)$ once and $f(0.5), \ldots, f(2.5)$ twice. B1 for seven ordinates. With 12 strips, $h = 3/12 = 0.25$ and 13 ordinates are needed, at $x = 0, 0.25, \ldots, 3$, of which the seven correct ones are every other one. B1 for 13 ordinates and $h = 0.25$. The strip width is $(b - a)/n$ with n the number of strips, so h was right; what goes wrong with six ordinates is the last term of the bracket, and with it the whole estimate. [3]

B4 TRA-B4

$h = 0.25$, and the ordinates at $x = 0, 0.25, 0.5, 0.75, 1$ are 1, 0.93941, 0.77880, 0.56978 and 0.36788. $L = 0.25(1 + 0.93941 + 0.77880 + 0.56978) = 0.25 \times 3.28799 =$ **0.8220** and $R = 0.25(0.93941 + 0.77880 + 0.56978 + 0.36788) = 0.25 \times 2.65587 =$ **0.6640**. M1 for the ordinates, A1 for L, A1 for R. On $[0, 1]$ the function decreases throughout, so on every strip the left-hand ordinate is the greater and the right-hand ordinate the lesser: L is the **upper** bound and R the **lower**, and decreasing on every strip is the condition that makes them so. B1 for the assignment with the condition. Adding the two sums counts each interior ordinate twice and each end ordinate once, which is the trapezium bracket, so $L + R = 2T$: $(0.8220 + 0.6640)/2 = 0.7430$, and $T = (0.25/2)[1 + 0.36788 + 2(0.93941 + 0.77880 + 0.56978)] = 0.7430$ confirms it. B1 for $L + R = 2T$. The true value is 0.7468, inside the bounds and within half their width, 0.079, of T. Here $f''(x) = (4x^2 - 2)e^{-x^2}$ changes sign at $x = 1/\sqrt{2}$, so the sign of f'' gives no direction, and it is the bounds that say anything certain. [5]

B5 TRA-B5 [4]

Seven readings, six strips, $h = 5$ s. $T = (5/2)[2.0 + 3.4 + 2(2.6 + 3.1 + 3.4 + 3.5 + 3.5)] = 2.5 \times 37.6 = 94$. M1 for the bracket with the readings as ordinates and $h = 5$, A1 for 94. Amperes multiplied by seconds are coulombs, so about **94** C passes, on the assumption that the current changes steadily between readings. B1 for the units with the assumption. The rectangle sums, $L = 5 \times 18.1 = 90.5$ C and $R = 5 \times 19.5 = 97.5$ C, bound the integral only where the current is monotonic on every strip. The readings themselves rise to 3.5 and then fall to 3.4, so the current turns somewhere in the last two intervals: before the turn the right-hand ordinate of a strip is the larger, after it the left-hand one is, and neither sum is on the same side of the area on every strip. B1 for the reason. Even a table that rose throughout would not show what happens inside its gaps. What can be reported is the estimate, about 94 C, and that readings taken more often would tighten it.

B6 TRA-B6 [6]

$R - L = h(y_1 + \dots + y_n) - h(y_0 + \dots + y_{n-1})$: every interior ordinate appears in both sums and cancels, leaving $h(y_n - y_0)$, the strip width times the total rise. M1 for the cancellation, A1 for the printed difference. Here $y_n - y_0 = 6$, so with $n = 6$ and $h = 0.5$ the bounds are **3** apart. B1 for 3. In general $h = 3/n$, so $R - L = 18/n$, and $18/n \le 0.1$ requires $n \ge 180$. M1 for the inequality, A1 for $\mathbf{n} = 180$. The trapezium estimate is the mean of the two bounds and the integral lies between them, so with 180 strips T is within **0.05** of the integral, half the width of the bounds. B1 for 0.05. The number of strips is settled before a single interior ordinate is computed, which is the use of the formula. It is a guarantee about the bounds: the trapezium estimate itself is usually far closer than 0.05.

B7 TRA-B7 [5]

$h = 0.5$, and the ordinates at $x = 0, 0.5, 1, 1.5, 2$ are 2, $\sqrt{3.5} = 1.870829$, $\sqrt{3} = 1.732051$, $\sqrt{2.5} = 1.581139$ and $\sqrt{2} = 1.414214$. $T = (0.5/2)[2 + 1.414214 + 2(1.870829 + 1.732051 + 1.581139)] = 0.25 \times 13.782252 = \mathbf{3.4456}$. M1 for the ordinates, A1 for 3.4456. With $f(x) = (4 - x)^{1/2}$, $f'(x) = -\tfrac{1}{2}(4 - x)^{-1/2}$ and $f''(x) = -\tfrac{1}{4}(4 - x)^{-3/2}$, negative on $[0, 2]$, so the curve is concave, each chord lies below it and the estimate is **below** the true value. M1 for f'', A1 for concave and therefore too small. Exactly, $\int(4 - x)^{1/2}\,dx = -(2/3)(4 - x)^{3/2}$, so the integral is $(2/3)(8 - 2\sqrt{2}) = 3.4477$ to 4 decimal places, about 0.002 above the estimate, as predicted. A1 for the exact value confirming the direction. Where an integral can be done exactly the trapezium rule is a check on the method rather than a route to the answer, and this one shows both the direction and the size of a four-strip error on a gently curved function.

B8 TRA-B8

The five ordinates are 0.5 apart, so $h = 0.5$ and there are four strips. The rule puts the two end ordinates in once and the three interior ones twice: $T = (0.5/2)[5.0 + 3.0 + 2(4.2 + 3.6 + 3.2)] = 0.25[8.0 + 2 \times 11.0] = 0.25 \times 30.0$. M1 for the bracket built that way with $h = 0.5$. So the estimate is **7.5**. A1 for 7.5. The first differences of the y values are −0.8, −0.6, −0.4 and −0.2, so the second differences are +0.2, +0.2 and +0.2. M1 for the two rounds of differences. They are positive, so the gradient is increasing, f'' is positive and the curve is **convex** across the whole interval. A1 for convex, with the positive second differences as the evidence. A chord joining two points of a convex curve lies above the curve between them, so each trapezium covers more than the strip beneath it and the estimate is an **overestimate**. B1 for the conclusion with the chord-above-the-curve reason. The values here fall while the curve stays convex, so a falling curve is no evidence either way: it is the second difference, not the first, that decides the direction of the error. [5]

SECTION C · Stretch

C1 TRA-C1

Seven readings, six strips of width $h = 10$ m, and the two end readings are 0 because the base line starts and ends at the water's edge. Area $\approx (10/2)[0 + 0 + 2(24 + 37 + 42 + 40 + 31)] = 5 \times 348 =$ **1740** m²: a width in metres times a distance in metres. B1 for $h = 10$ with the seven widths as ordinates, M1 for the bracket, A1 for 1740 m². Volume $\approx 1740 \times 2.5 =$ **4350** m³, about 4400 m³ to the precision the survey supports. A1 for the volume with its units. The area assumes the edge of the pond runs straight between consecutive measured widths, and the volume assumes the depth really is 2.5 m everywhere. B1 for both assumptions. Nothing was measured between the survey lines, so the table does not fix the curve of the shoreline there: a bay between two lines would make the estimate too large and a bulge too small, and the pattern of a finite table, whatever it suggests, is not the sign of f'' on the strips. Nor can the rectangle sums be offered as bounds, because the widths rise and then fall. B1 for the reason. More closely spaced survey lines would improve the estimate; nothing in this table can say in which direction. [6]

C2 TRA-C2

$h = 0.5$, and the ordinates at $x = 0, 0.5, 1, 1.5, 2$ are 1, 0.8, 0.5, 0.30769 and 0.2. $T = (0.5/2)[1 + 0.2 + 2(0.8 + 0.5 + 0.30769)] = 0.25 \times 4.41538 =$ **1.1038**. M1 for the ordinates, A1 for 1.1038. Differentiating twice, $f'(x) = -2x(1 + x^2)^{-2}$ and $f''(x) = (6x^2 - 2)/(1 + x^2)^3$, which is negative for x below $1/\sqrt{3} = 0.577$ and positive beyond it. M1 for f'' with its sign change. The sign at one point says nothing about the other strips: on [0, 0.5] the curve is concave and the chord lies below it, on [1, 1.5] and [1.5, 2] it is convex and the chords lie above, and the strip [0.5, 1] contains the change. The errors have opposite signs, so no direction for the total follows from f''. A1 for the explanation. Against the exact 1.1071, the estimate 1.1038 is in fact **too** small: the first strip, where the curve bends most sharply, loses more than the gently convex later strips gain. B1 for the comparison. A direction can be claimed only when f'' keeps one sign on every strip, and this integral is the case the lesson warns about. [5]

C3 TRA-C3

$h = 1$, and the ordinates at $x = 0, 1, 2, 3$ are 0, $e^{-1} = 0.36788$, $2e^{-2} = 0.27067$ and $3e^{-3} = 0.14936$. $L = 1(0 + 0.36788 + 0.27067) =$ **0.6386** and $R = 1(0.36788 + 0.27067 + 0.14936) =$ **0.7879**. M1 for the ordinates and the two sums, A1 for both values. Both lie below 0.8009, so neither is an upper bound. The reason is that $f'(x) = (1 - x)e^{-x}$ changes sign at $x = 1$: f increases on the first strip and decreases on the other two, so the right-hand ordinate is the larger on [0, 1] but the smaller on [1, 2] and [2, 3], and R mixes one overcount with two undercounts. B1 for the turning point at $x = 1$ and what it does to the sums. Take, strip by strip, the larger ordinate for an upper bound and the smaller for a lower bound: upper $1(0.36788 + 0.36788 + 0.27067) =$ **1.0064**, lower $1(0 + 0.27067 + 0.14936) =$ **0.4200**. M1 for choosing the extreme ordinate on each strip, A1 for both bounds. The integral, 0.8009, lies between them, as it must. The bounds are wide because $h = 1$ is coarse and the function's whole rise and fall sits inside three strips. Splitting at the turning point is what makes each piece monotonic, and it needs no extra ordinate here because $x = 1$ is already a strip boundary. [5]

C4 TRA-C4

Seven readings, six strips, $h = 15$ min. $T = (15/2)[3.2 + 1.6 + 2(2.7 + 2.3 + 2.0 + 1.8 + 1.7)] = 7.5 \times 25.8 = 193.5$, and litres per minute multiplied by minutes are litres, so about **190** litres to the precision the readings support, assuming the rate changes steadily between readings. M1 for the bracket with $h = 15$, A1 for 193.5 with the units, B1 for the assumption. The engineer's claim needs the rate to be decreasing throughout every 15-minute interval, not only at the seven instants recorded: a table cannot show what happens inside its gaps. B1 for the monotonic assumption stated about the intervals. With that assumption, $L = 15(3.2 + 2.7 + 2.3 + 2.0 + 1.8 + 1.7) = 15 \times 13.7 =$ **205.5** litres is the upper bound and $R = 15 \times 12.1 =$ **181.5** litres the lower, 24 litres apart, which is $h(y_0 - y_n) = 15 \times 1.6$. A1 for the two sums, correctly assigned. The estimate is improved by reading the rate more often: with $h = 5$ the bounds would be $5 \times 1.6 =$ **8** litres apart, a third of the present gap, and the trapezium estimate would then be within 4 litres of the true volume rather than 12. B1 for more frequent readings with the gap of 8 litres. [6]

THIS BOOK ONLINE

Scan the code, or type the address, to open the questions in this book online:

inklearning.co.uk/maths/practise/questions/the-trapezium-rule/