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ANSWER BOOK

The vector product and the scalar triple product

Further Pure 1 · Year FM

7 questions · 25 marks

NAVY EDITION

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SECTION A · Warm-up

A1 $|a \times b|$ is the area of the parallelogram spanned by a and b, equal to $|a||b| \sin \theta$. The scalar triple product $a \cdot (b \times c)$ is the volume of the parallelepiped on the three vectors, read as a modulus, since the triple product itself can come out negative. B1 for the area of the parallelogram, B1 for the volume of the parallelepiped. [2]

A2 $a \times b = ((-1)(-2) - (3)(4), (3)(1) - (2)(-2), (2)(4) - (-1)(1)) = $ **(-10,** 7, 9). Dot with a: $-20 - 7 + 27 =$ **0**, so the product is perpendicular to a, as it must be. M1 for the vector product, A1 for $(-10, 7, 9)$, B1 for the zero dot product. Dotting with b gives 0 as well. The commonest slip is the sign of the middle component, and this check catches it. [3]

SECTION B · Standard

B1 Work with edge vectors from a common vertex: $AB = (2, -1, 3)$ and $AC = (1, 4, -2)$. Their vector product is **(-10,** 7, 9), of magnitude $\sqrt{(100 + 49 + 81)} = \sqrt{230} \approx 15.17$, the area of the parallelogram on AB and AC. The triangle is half of it, so its area is **$\sqrt{230}/2$** ≈ 7.58. M1 for the edge vectors, M1 for the vector product, A1 for $\sqrt{230}$, A1 for half of it. Crossing the position vectors instead of the differences is the usual error and loses the first method mark. [4]

B2 $b \times c = ((3)(4) - (1)(2), (1)(0) - (2)(4), (2)(2) - (3)(0)) = (10, -8, 4)$. Then $a \cdot (b \times c) = 10 - 8 + 4 = 6$. [4]
The parallelepiped's volume is the **modulus** of that, $|6| = 6$, and the tetrahedron takes a sixth of it, so its volume is **1**. M1 for $b \times c$, A1 for $(10, -8, 4)$, M1 for the scalar triple product, A1 for the volume. Take the modulus even when the answer is already positive. Listing the same three vectors in the other order gives -6, and a volume is never negative.

B3 Taking a as the position vector of A, the equation is $(\mathbf{r} - (2i - j + 4k)) \times (i + 2j + 2k) = 0$. Every point of l gives a displacement from A parallel to b, and the vector product of parallel vectors is zero. [3]
$|b| = \sqrt{(1 + 4 + 4)} = 3$, so the direction cosines are **1/3,** 2/3 and 2/3. B1 for the vector form of the line, M1 for $|b| = 3$, A1 for the direction cosines. Check: $(1 + 4 + 4)/9 = 1$, which the cosines of the angles a direction makes with the three axes must always satisfy.

B4 A zero triple product means the parallelepiped has no volume, so the three vectors are **coplanar**: one lies in the plane of the other two. A zero vector product means the parallelogram has no area, so those two vectors are **parallel**. B1 for coplanar, B1 for parallel, B1 for the comparison. Both are degeneracy tests, one dimension apart. [3]

SECTION C · Stretch

C1 The direction vectors are $d_1 = (2, 1, -1)$ and $d_2 = (1, 2, 1)$. Neither is a multiple of the other, so the lines are not parallel. [6]

$d_1 \times d_2 = ((1)(1) - (-1)(2), (-1)(1) - (2)(1), (2)(2) - (1)(1)) = \mathbf{(3, -3, 3)}$, of magnitude $\sqrt{(9 + 9 + 9)} = \mathbf{3\sqrt{3}}$.

Join a point of each line: $a_2 - a_1 = (3, -1, 0) - (1, 0, 2) = (2, -1, -2)$. Then $(a_2 - a_1)\cdot(d_1 \times d_2) = 6 + 3 - 6 = \mathbf{3}$. That is non-zero, so the lines do not intersect. Not parallel and not intersecting makes them **skew**.

The shortest distance is the component of $a_2 - a_1$ along the common perpendicular: $|(a_2 - a_1)\cdot(d_1 \times d_2)|/|d_1 \times d_2| = 3/(3\sqrt{3}) = 1/\sqrt{3} = \mathbf{\sqrt{3}/3}$, about 0.577.

B1 for the directions not being parallel, M1 for $d_1 \times d_2$, A1 for $(3, -3, 3)$, M1 for the scalar triple product, A1 for the skew conclusion, A1 for the distance.

Two marks hang on the modulus. The triple product carries the sign of the orientation, so naming the lines the other way round turns 3 into -3, and a distance is never negative. One triple product does both jobs here: non-zero proves the lines are skew, and its size gives the gap.