inkMaths

ANSWER BOOK

Transformations of the complex plane

Further Pure 2 · Year FM

6 questions · 26 marks

NAVY EDITION

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SECTION A · Warm-up

A1 The transformation is a translation by the vector $2 - i$, so $z = w - 2 + i$. [3]
Substituting gives $|w - 2 + i| = 3$, a circle of the same radius 3, now centred at $2 - i$. B1 for recognising a translation, M1 for $z = w - 2 + i$, A1 for the image equation. Translations move a locus without changing its shape or size.

A2 Since $2i$ has modulus 2 and argument $\pi/2$, the map is an enlargement scale factor 2 about the origin combined with a rotation of 90° anticlockwise. Substituting $z = w/(2i)$ gives $|w/(2i) - 1| = 1$, and multiplying through by $|2i| = 2$ gives $|w - 2i| = 2$. B1 for the enlargement, B1 for the rotation, M1 for substituting for z, A1 for the image circle. The centre 1 has moved to $2i$ and the radius has doubled. [4]

SECTION B · Standard

B1 Write $w = u + iv$, so $z = 1/w = (u - iv)/(u^2 + v^2)$ and $\mathrm{Re}(z) = u/(u^2 + v^2)$. Setting that equal to $1/2$ gives $u^2 + v^2 = 2u$, that is $(u - 1)^2 + v^2 = 1$: a circle of centre 1 and radius 1. M1 for $z = 1/w$, A1 for $\mathrm{Re}(z) = u/(u^2 + v^2)$, M1 for $u^2 + v^2 = 2u$, A1 for the circle equation, B1 for the centre and radius. The image passes through the origin, which is the image of the point at infinity along the line. [5]

B2 Taking moduli, $|w| = 1/|z| = 1/2$, so the image is the circle $|w| = 1/2$. Since $\arg w = -\arg z$, a point moving anticlockwise round the original circle traces the image clockwise. M1 for taking moduli, A1 for $|w| = 1/2$, M1 for $\arg w = -\arg z$, A1 for the reversed sense. Inversion reflects in the real axis as well as reciprocating the modulus. [4]

B3 Put $z = 1 + it$, so $w = 1 - t^2 + 2it$. Then $u = 1 - t^2$ and $v = 2t$, so $t = v/2$ and $u = 1 - v^2/4$. Rearranged, $v^2 = 4(1 - u)$: a parabola with vertex at $u = 1$ opening in the negative u direction. M1 for $z = 1 + it$, A1 for $u = 1 - t^2$ and $v = 2t$, M1 for eliminating t, A1 for $v^2 = 4(1 - u)$. Squaring does not keep straight lines straight. [4]

SECTION C · Stretch

C1 Put $z = \cos\theta + i\sin\theta$. Then $w = (\cos\theta + 1 + i\sin\theta)/(\cos\theta - 1 + i\sin\theta)$; multiplying top and bottom by the conjugate of the denominator makes the denominator $(\cos\theta - 1)^2 + \sin^2\theta = 2 - 2\cos\theta$, and the real part of the numerator is $(\cos^2\theta - 1) + \sin^2\theta = 0$. So w is purely imaginary for every θ. The point $z = 1$ is excluded, since the denominator vanishes there, and no finite w corresponds to it. M1 for $z = \cos\theta + i\sin\theta$, M1 for multiplying by the conjugate, A1 for the denominator, M1 for the real part of the numerator, A1 for showing it vanishes, B1 for excluding $z = 1$. [6]